

# Trigonometric Functions, Exponential Functions, Complex Numbers, Integration and Differentiation Summary

Ánoq of the Sun, Hardcore Processing \*

June 30, 2004

## Basic Complex Numbers

**Premises:**  $r, \theta, a, b \in \mathbb{R}, z \in \mathbb{C}, z = a + ib = r \cdot e^{i\theta}$

$a + bi$	(rectangular form)	p. 248 [3]
$r \cdot e^{i\theta}$	(polar form)	p. 248 [3]
$\operatorname{Re}(a + bi) = a$	(real part)	p. 3 [4]
$\operatorname{Im}(a + bi) = b$	(imaginary part)	p. 3 [4]
$r =  a + bi  = \sqrt{a^2 + b^2}$	(numeric value - a.k.a. modulus)	p. 3 [4]
$\forall r \neq 0 : \operatorname{arg}(r \cdot e^{i\theta}) = \theta$	(argument)	p. 3 [4]
$a = r \cdot \cos \theta$	(rectangular vs. polar conversion)	p. 248 [3]
$b = r \cdot \sin \theta$	(rectangular vs. polar conversion)	p. 248 [3]
$a + ib = r(\cos \theta + i \sin \theta)$	(rectangular vs. polar conversion)	p. 3 [4]
$r = \sqrt{a^2 + b^2}$	(rectangular vs. polar conversion)	p. 248 [3]
$i^2 = -1$	(i)	p. 252 [3]

## Algebra in $\mathbb{C}$

$z + w = w + z$	$z \cdot w = w \cdot z$	(commutative)	p. 250 [3]
$z + (w + u) = (z + w) + u$	$z \cdot (w \cdot u) = (z \cdot w) \cdot u$	(associative)	p. 250 [3]
$z \cdot (w + u) = z \cdot w + z \cdot u$	$z \cdot (w - u) = z \cdot w - z \cdot u$	(distributive)	p. 250 [3]
$z + 0 = z$	$z \cdot 1 = z$	(0 and 1)	p. 250 [3]
$z - z = 0$	$z \cdot (1/z) = 1, z \neq 0$	(opposite numbers)	p. 250 [3]
$z - w = z + (-w)$	$z/w = z \cdot (1/w)$	(minus and division)	p. 250 [3]

## Complex Conjugate (only in $\mathbb{R}$ ?)

$\overline{z} \cdot \overline{w} = \overline{zw}$	$\frac{\overline{z}}{\overline{w}} = \overline{\left(\frac{z}{w}\right)}, w \neq 0$	(multiply and divide)	p. 253 [3]
$\overline{z} + \overline{w} = \overline{z + w}$	$\overline{z} - \overline{w} = \overline{z - w}$	(plus and minus)	p. 253 [3]
$z \cdot \overline{z} =  z ^2$			p. 253 [3]

---

\*© 2001 Ánoq of the Sun (alias Johnny Andersen)

## Misc

- A complex number is rational if both real and imaginary parts are rational.  
(p. 1.16 [1])
- $f(\theta) = e^{i\theta}$  defines a *continuous group homomorphism* of the group of the real numbers  $(\mathbb{R}, +)$  on the circle group  $(\mathbb{T}, \cdot)$ , where  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$   
(p. 22 [4])

## Exponential function ( $\mathbb{C}$ )

**Premises:**  $a, b, r, s, \theta, \phi \in \mathbb{R}, n \in \mathbb{Z}$

$e^{a+bi} \stackrel{\text{def}}{=} e^a e^{bi}$	(complex exponent def.)	p. 251 [3]
$(re^{i\theta})(se^{i\phi}) = (rs)e^{i(\theta+\phi)}$	(multiply power rule)	p. 251 [3]
$e^{a+ib} = e^a(\cos b + i \sin b)$		p. 22 [4]
$ e^{a+ib}  = e^a$	(radius of exponent)	rem. 1.17, p. 22 [4]
$\operatorname{Re}(e^{a+ib}) = e^a \cos b$	(real part of exponent)	rem. 1.17, p. 22 [4]
$\operatorname{Im}(e^{a+ib}) = e^a \sin b$	(imaginary part of exponent)	rem. 1.17, p. 22 [4]
$(\cos \theta + i \sin \theta)^\phi = \cos(\theta\phi) + i \sin(\theta\phi)$	(De Moivre)	p. 22 [4] (only shown for $\mathbb{N}$ )
$e^{i(\pi n)} = e^{i(-\pi n)}$	(n full circles on $\mathbb{C}$ )	

**Premises:**  $x, y, z \in \mathbb{C}, n \in \mathbb{Z}$

$e^{ix} = (\cos x) + (\sin x)i$	(Euler)	p. 21 [4], p. 249 [3]
$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$	(Euler)	p. 21 [4], p. 2.1 [1]
$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$	(Euler)	p. 21 [4], p. 2.1 [1]
$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$	(definition)	(p. 23 [4], p. 122 [3])
$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$	(definition)	(p. 23 [4], p. 122 [3])
$e^{x+y} = e^x e^y$	(exponent rule)	p. 20 [4]
$e^{-x} = \frac{1}{e^x}$	(negative exponent)	p. 20 [4]
$(e^x)^y = e^{xy}$		
$(xe^y)^z = x^z e^{yz}$	(exponent distrib)	
$x = e^y = e^{i2\pi n}$	(e is $2\pi i$ -periodic)	(p. 22 [4])
$e^x = 1 \Leftrightarrow x \in 2\pi i\mathbb{Z}$	(1-crossings for exp)	thm. 1.18, p. 23 [4]
$\cos x = 0 \Leftrightarrow x \in \{\frac{\pi}{2} + \pi\mathbb{Z}\}$	(0-crossings for cos)	thm. 1.18, p. 23 [4]
$\sin x = 0 \Leftrightarrow x \in \pi\mathbb{Z}$	(0-crossings for sin)	thm. 1.18, p. 23 [4]
$\cosh x = 0 \Leftrightarrow x \in i\{\frac{\pi}{2} + \pi\mathbb{Z}\}$ , for $\cosh : \mathbb{C} \rightarrow \mathbb{C}$	(0-crossings for cosh)	(p. 24 [4])
$\sinh x = 0 \Leftrightarrow x \in i\pi\mathbb{Z}$ , for $\sinh : \mathbb{C} \rightarrow \mathbb{C}$	(0-crossings for sinh)	(p. 24 [4])

## Square Root

$\forall \theta \in [0, 2\pi[ : \forall r \geq 0 \in \mathbb{R} : \forall z \in \mathbb{C}, z = r \cdot e^{i\theta} : \sqrt{z} \doteq \sqrt{r} \cdot e^{i(\theta/2)}$	def p. 254 [3]
$\forall x, z \in \mathbb{C} : w^2 = z \Leftrightarrow w = \pm\sqrt{z}$	thm. 9.4 p. 254 [3]
$\forall x \in \mathbb{R}, x > 0 : \sqrt{-x} = \sqrt{(-1) \cdot x} = \sqrt{-1} \cdot \sqrt{x} = i\sqrt{x}$	thm. 9.5 p. 255 [3]
Be careful: $\sqrt{z \cdot w} \neq \sqrt{z} \cdot \sqrt{w} !!$	p. 255 [3]

## Metric stuff in $\mathbb{C}$

$d(z, w) =  z - w $	(metric - like $\mathbb{R}^2$ euclidean dist.)	p. 5 [4]
<i>open, closed, limited</i> sets are like $\mathbb{R}^2$		p. 5 [4]
$K(a, r) = \{z \in \mathbb{C} \mid  a - z  < r\}$	(sphere)	p. 5 [4]
$K'(a, r) = K(a, r) \setminus \{a\} = \{z \in \mathbb{C} \mid 0 <  a - z  < r\}$	(sphere excluding center)	p. 5 [4]
$f : A \rightarrow \mathbb{C}$ for $A \subseteq \mathbb{C}$	(complex function)	p. 5 [4]
$(\operatorname{Re} f)(z) = \operatorname{Re} (f(z)), (\operatorname{Im} f)(z) = \operatorname{Im} (f(z))$	(real and im. function parts)	p. 5 [4]
$f(z) = \operatorname{Re} f(z) + i \operatorname{Im} f(z)$ for $z \in A$	(real and im. function parts)	p. 6 [4]
$ \operatorname{Re} z  \leq  z ,  \operatorname{Im} z  \leq  z ,  z  \leq  \operatorname{Re} z  +  \operatorname{Im} z , z \in \mathbb{C}$	(Re and Im vs. z bounds)	p. 6 [4]
$\forall \epsilon > 0 \exists \delta > 0 \forall z \in A :  z - z_0  < \delta \Rightarrow  f(z) - f(z_0)  < \epsilon$	(continuous function)	p. 5 [4]
$\operatorname{Re} (f)$ and $\operatorname{Im} (f)$ <i>continuous</i> $\Leftrightarrow f$ <i>continuous</i>	(continuous parts)	p. 6 [4]

## Formulas for complex numbers

$\forall x, y \in \mathbb{C}, n \in \mathbb{N} :$

$ x + y ^2 \leq 2( x ^2 +  y ^2)$ (Cauchy-Schwartz for $(1, 1)$ and $(x, y)$ in $\mathbb{C}^2$ )	p. 1.10 [1]
$\operatorname{Re} x \leq  \operatorname{Re} x  \leq  x $	p. 30 [4]
$ xy  \leq \frac{1}{2}( x ^2 +  y ^2)$	p. 1.10 [1]
$(x^n - y^n)/(x - y) = \sum_{k=1}^n x^{n-k} y^{k-1}$	p. 18 [4]

# 1 Trigonometry

## 1.1 Definitions of Trigonometric Functions ( $\mathbb{C}$ )

$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots$	(p. 19 [4])
$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots$	(p. 21 [4])
$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots$	(p. 21 [4])
$\tan(x) = \frac{\sin(x)}{\cos(x)}$ , for $x \in \mathbb{C} \setminus \{\frac{\pi}{2} + \pi\mathbb{Z}\}$	(p. 23 [4], p. 18 [2], p. 118 [3])
$\cot(x) = \frac{\cos(x)}{\sin(x)}$ , for $x \in \mathbb{C} \setminus \pi\mathbb{Z}$	(p. 23 [4], p. 118 [3])
$\sec(x) = \frac{1}{\cos(x)}$	Internet.. (only $\mathbb{R}$ ?)
$\csc(x) = \frac{1}{\sin(x)}$	Internet.. (only $\mathbb{R}$ ?)
$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$	(p. 23 [4])
$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$	(p. 23 [4])
$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$	(p. 23 [4], p. 122 [3])
$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$	(p. 23 [4], p. 122 [3])
$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ , for $x \in \mathbb{C} \setminus i\{\frac{\pi}{2} + \pi\mathbb{Z}\}$	(p. 23 [4], p. 122 [3])
$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$ , for $x \in \mathbb{C} \setminus i\pi\mathbb{Z}$	(p. 23 [4], p. 122 [3])
$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$	Internet.. (only $\mathbb{R}$ ?)
$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$	Internet.. (only $\mathbb{R}$ ?)

## 1.2 Identities

### 1.2.1 Exponential Identities ( $\mathbb{C}$ )

$\exp(x + y) = \exp(x) \exp(y)$	(p. 20 [4])
$e = \exp(1) \approx 2.718\dots$	(p. 20 [4])
$\exp(x) = e^x$	(p. 21 [4])

### 1.2.2 Sine / Cosine Identities

$$\begin{aligned}
 \cos^2 x + \sin^2 x &= 1, \text{ for } x \in \mathbb{C} && \text{(idiot formula, p. 24 [4], p. 17 [2])} \\
 \cos(x + 2\pi) &= \cos(x) && \text{(p. 17 [2])} \\
 \sin(x + 2\pi) &= \sin(x) && \text{(p. 17 [2])} \\
 \cos(-x) &= \cos(x) && \text{(p. 21 [4], p. 17 [2])} \\
 \sin(-x) &= -\sin(x) && \text{(p. 21 [4], p. 17 [2])} \\
 \cos(\pi - x) &= -\cos(x) && \text{(p. 17 [2])} \\
 \sin(\pi - x) &= \sin(x) && \text{(p. 17 [2])} \\
 \cos\left(\frac{\pi}{2} - x\right) &= \sin(x) && \text{(p. 17 [2], p. 116 [3])} \\
 \sin\left(\frac{\pi}{2} - x\right) &= \cos(x) && \text{(p. 17 [2], p. 116 [3])} \\
 \cos(x + y) &= \cos(x)\cos(y) - \sin(x)\sin(y), \text{ for } x, y \in \mathbb{C} && \text{(p. 22 [4], p. 116 [3])} \\
 \sin(x + y) &= \sin(x)\cos(y) + \cos(x)\sin(y), \text{ for } x, y \in \mathbb{C} && \text{(p. 22 [4], p. 116 [3])} \\
 \cos(x - y) &= \cos(x)\cos(y) + \sin(x)\sin(y) && \text{(p. 116 [3])} \\
 \sin(x - y) &= \sin(x)\cos(y) - \cos(x)\sin(y) && \text{(p. 116 [3])} \\
 \cos(2x) &= \cos^2(x) - \sin^2(x) && \text{(p. 116 [3])} \\
 &= 1 - 2\sin^2(x) \\
 &= 2\cos^2(x) - 1 \\
 \sin(2x) &= 2\sin(x)\cos(x) && \text{(p. 116 [3])} \\
 \cos^2\left(\frac{x}{2}\right) &= \frac{1}{2}(1 + \cos x) \\
 \cos x = 0 &\Leftrightarrow x \in \left\{\frac{\pi}{2} + \pi\mathbb{Z}\right\} \text{ (0-crossings for cos)} && \text{thm. 1.18, p. 23 [4]} \\
 \sin x = 0 &\Leftrightarrow x \in \pi\mathbb{Z} \text{ (0-crossings for sin)} && \text{thm. 1.18, p. 23 [4]}
 \end{aligned}$$

### 1.2.3 Tangent etc. Identities

$$\begin{aligned}
 \tan(-x) &= -\tan(x) && \text{(p. 18 [2])} \\
 \tan(x + \pi) &= \tan(x) && \text{(p. 18 [2])} \\
 \tan\left(\frac{1}{2}x\right) &= \frac{\sin(x)}{1 + \cos(x)} && \text{(Mat3GE instructor)} \\
 \tan^2(x) + 1 &= \sec^2(x) = \frac{1}{\cos^2(x)} && \text{(p. 19, 30 [2])} \\
 \sec^2(x) &= \tan^2(x) + 1 && \text{(p. 19, 30 [2])}
 \end{aligned}$$

### 1.2.4 Hyperbolic Identities

$$\begin{aligned}
 (\cosh x)^2 + (i \sinh x)^2 &= 1, \text{ for } x \in \mathbb{C} && \text{(p. 24 [4])} \\
 \cosh^2 x - \sinh^2 x &= 1, \text{ for } x \in \mathbb{C} && \text{(p. 24 [4], p. 122 [3])} \\
 \cosh(ix) &= \cos x, \text{ for } x \in \mathbb{C} && \text{(p. 24 [4])} \\
 \sinh(ix) &= i \sin x, \text{ for } x \in \mathbb{C} && \text{(p. 24 [4])} \\
 \cos(ix) &= \cosh x, \text{ for } x \in \mathbb{C} && \text{(p. 24 [4])} \\
 \sin(ix) &= i \sinh x, \text{ for } x \in \mathbb{C} && \text{(p. 24 [4])} \\
 \cosh(x + y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) && \text{(p. 122 [3])} \\
 \sinh(x + y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y) && \text{(p. 122 [3])} \\
 \sinh(2x) &= 2 \sinh(x)\cosh(x) && \text{(p. 122 [3])} \\
 \cosh(2x) &= \cosh^2(x) + \sinh^2(x) && \text{(p. 122 [3])} \\
 &= 1 + 2\sinh^2(x) \\
 &= 2\cosh^2(x) - 1 \\
 \cosh x = 0 &\Leftrightarrow x \in i\left\{\frac{\pi}{2} + \pi\mathbb{Z}\right\}, \text{ for } \cosh : \mathbb{C} \rightarrow \mathbb{C} && \text{(p. 24 [4])} \\
 \sinh x = 0 &\Leftrightarrow x \in i\pi\mathbb{Z}, \text{ for } \sinh : \mathbb{C} \rightarrow \mathbb{C} && \text{(p. 24 [4])}
 \end{aligned}$$

## 2 Differentiation and Integration Rules

$\phi(x)$	$\phi'(x) \equiv \frac{d}{dx}\phi(x)$	$\int \phi(x)dx$	Ref.
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$	$F(x)g(x) - \int F(x)g'(x)dx$ where $F(x) = \int f(x)dx$	(p. 10, 31, Integration by Parts [2])
$f(g(x))$	$f'(g(x))g'(x)$	$\int f(t)dt$ , where $t = g(x)$	(p. 10, 31, Substitution [2])
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$		(p. 10 [2])

## 3 Differentiation and Integration Table ( $\mathbb{C}$ )

$f(x)$	$f'(x) \equiv \frac{d}{dx}f(x)$	$\int f(x)dx$	Ref.
$e^x, x \in \mathbb{C}$	$e^x$	(Tmp: formula below for $\mathbb{C}$ ?)	(p. 19 [4])
$e^{kx}$	$ke^{kx}$	$\frac{1}{k}e^{kx} + C$	(p. 14, 30 [2])
$\cos x$	$-\sin x$	$\sin x + C$	(p. 21 [4], p. 19 [2])
$\sin x$	$\cos x$	$-\cos x + C$	(p. 21 [4], p. 19 [2])
$\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$	$-\ln( \cos(x) ) + C$	(exc. 1.17 [4], p. 19, 30 [2])
$\cosh(x)$	$\sinh(x)$	$\sinh(x) + C$	(p. 24 [4], p. 122 [3])
$\sinh(x)$	$\cosh(x)$	$\cosh(x) + C$	(p. 24 [4], p. 122 [3])

## 4 Differentiation and Integration Table ( $\mathbb{R}$ )

$f(x)$	$f'(x) \equiv \frac{d}{dx}f(x)$	$\int f(x)dx$	Ref.
$\cot x$	$-\csc^2(x)$	$\ln( \sin(x) ) + C$	Internet...
$\sec(x)$	$\sec(x)\tan(x)$	$\ln( \sec(x) + \tan(x) ) + C$	Internet...
$\csc(x)$	$-\csc(x)\cot(x)$	$\ln( \csc(x) + \cot(x) ) + C$	Internet...
$\tanh(x)$	$\frac{1}{\cosh^2(x)}$		(p. 122 [3])
$\coth(x)$	$-\frac{1}{\sinh^2(x)}$		(p. 122 [3])

### 4.1 Composite Differentiation and Integration

$f(x)$	$f'(x) \equiv \frac{d}{dx}f(x)$	$\int f(x)dx$	Ref.
$\cos(kx)$	$-kx \sin(kx)$		
$\sin(kx)$	$kx \cos(kx)$		
$\cosh^2(x)$	$2 \sinh(x) \cosh(x)$		
$\sinh^2(x)$	$2 \sinh(x) \cosh(x)$		

## 5 Integration Identities ( $\mathbb{C}$ )

For  $f, g : [a, b] \rightarrow \mathbb{C}$ ,  $f = \text{Re } f + i \text{Im } f$ :

- $\int_a^b f(x)dx \stackrel{\text{def}}{=} \int_a^b \text{Re } f(x)dx + i \int_a^b \text{Im } f(x)dx$  (p. 29  $\in$  [4]) (1.3  $\in$  [1])
- $\text{Re}(\int_a^b f(t)dt) = \int_a^b \text{Re } f(t)dt$  (p. 29  $\in$  [4])
- $\text{Im}(\int_a^b f(t)dt) = \int_a^b \text{Im } f(t)dt$  (p. 29  $\in$  [4])
- $\int_a^b (f(t) + g(t))dt = \int_a^b f(t)dt + \int_a^b g(t)dt$  (p. 29  $\in$  [4])
- $\int_a^b cf(t)dt = c \int_a^b f(t)dt$  (p. 29  $\in$  [4])

- $\int_a^b \overline{f(x)} dx = \overline{\int_a^b f(x) dx}$  (complex conjugate) (1.3 ∈ [1])
- $|\int_a^b f(t) dt| \leq \int_a^b |f(t)| dt$  (thm. 2.1 p. 29 ∈ [4])

## References

- [1] Bergfinnur Durhuus. *Hilbert Rum med Anvendelser*, Matematisk Afdeling Københavns Universitet 1997.
- [2] *Matematisk Formelsamling*, Undervisningsministeriet, Gymnasieafdelingen 1993.
- [3] Arne Hole. *Klassisk Analyse og Lineær Algebra*, Universitetsforlaget 1998.
- [4] Christian Berg. *Kompleks Funktionsteori*, Matematisk Institut Københavns Universitet 2002.