Topology Summary

Ánoq of the Sun, Hardcore Processing *

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1 Topology - The Creation

Premises: A, B ***sets***

- **A *topology* on** $A \stackrel{def}{=}$ a *collection* $\mathcal{T}$ of *subsets* of A where:
  1) $\emptyset, A \in \mathcal{T}$
  2) For *arbitrary* index set $I : (\forall i \in I : A_i \in \mathcal{T}) \Rightarrow \bigcup_{i \in I} A_i \in \mathcal{T}$
  3) For *finite* index set $I : (\forall i \in I : A_i \in \mathcal{T}) \Rightarrow \bigcap_{i \in I} A_i \in \mathcal{T}$
  ⟨p. 76 ∈ [1]⟩

- **A *toplogogical space*** $(A, \mathcal{T})$ **with *topology*** $\mathcal{T} \stackrel{def}{=}$
  a set A for which a *topology* $\mathcal{T}$ has been specified. ⟨p. 76 ∈ [1]⟩

- For *topological space* A with *topology* $\mathcal{T}$:
  U **is an *open set* of** $A \stackrel{def}{=} U \subset A$ and $U \in \mathcal{T}$ ⟨p. 76 ∈ [1]⟩
  U **is a *neighborhood* of** $x \stackrel{def}{=} U$ is an *open set containing* x (i.e. U *open* and $x \in U$) ⟨p. 96 ∈ [1]⟩
  Warning: Some people say that U is a neighborhood of x if
  U *contains* an open set V, where $x \in V$ (i.e. $\exists V \subset U : x \in V$) ⟨p. 97 ∈ [1]⟩

1.1 Bases

- **A *basis*** $\mathcal{B}$ **for a *topology* on** $A \stackrel{def}{=}$
  a collection $\mathcal{B}$ of subsets of A (called the *basis elements*) such that:
  1) $\forall a \in A : \exists B \in \mathcal{B} : a \in B$
  2) $\forall B_1, B_2 \in \mathcal{B} : a \in B_1 \cap B_2 \Rightarrow \exists B_3 \in \mathcal{B} : a \in B_3 \subset (B_1 \cap B_2)$
  ⟨p. 78 ∈ [1]⟩
  The *basis* of a *given topology* is *not necessarily unique*. ⟨p. 80 ∈ [1]⟩

- **The *topology*** $\mathcal{T}$ ***generated* by the *topology basis*** $\mathcal{B}$:
  $\mathcal{T} \stackrel{def}{=}$ all $U \subset A$ such that: $\forall a \in U : \exists B \in \mathcal{B} : a \in B \subset U$. ⟨p. 78 ∈ [1]⟩
  Prooving: Proove each of 1), 2), 3) in def. of topology separately like this: Use definition of generated topology and use the rules 1), 2) from topology basis as needed.

- $\mathcal{B}$ is a *basis* for the topology $\mathcal{T} \Rightarrow$
  $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$. ⟨lemma 13.1 p. 80 ∈ [1]⟩

- Let $(A, \mathcal{T})$ be a *given topological space*.
  If $\mathcal{C} \subset \mathcal{T}$ such that: $\forall U \in \mathcal{T} : \forall x \in U : \exists C \in \mathcal{C} : x \in C \subset U$,
  then $\mathcal{C}$ is a *basis* for $\mathcal{T}$. ⟨lemma 13.2 p. 80 ∈ [1]⟩

1.2 Subbases

- **A *subbasis* $\mathcal{S}$ for a *topology* on** $A \overset{def}{=}$
  a collection $\mathcal{S}$ of subsets of A, where $A = \bigcup_{S \in \mathcal{S}} S$ ⟨p. 82 ∈ [1]⟩
  (The *subbasis* of a *given topology* is *not necessarily unique*?)

- **The *topology* $\mathcal{T}$ *generated* by the *topology subbasis* $\mathcal{S}$:**
  $\mathcal{T} \overset{def}{=}$ *all unions* of *finite intersections* of elements of $\mathcal{S}$. ⟨p. 82 ∈ [1]⟩
  I.e. all $U \subset A$ such that: $\exists S_{ij} \in \mathcal{S} : U = \bigcup_{i \in I} \bigcap_{j \in J} S_{ij}$ for *finite* J.

2 Common and Well-Known

2.1 Common Well-Known Topologies

Premises: A, B ***sets***

- ***Discrete*** **topology on** $A \overset{def}{=} \mathcal{T}$ is *all subsets* of A ⟨ex. 2 p. 77 ∈ [1]⟩
- ***Indiscrete*** / ***trivial*** **topology on** $A \overset{def}{=} \mathcal{T} = \{\emptyset, A\}$ ⟨ex. 2 p. 77 ∈ [1]⟩
- ***Finite complement*** **topology** $\mathcal{T}_f$ **on** $A \overset{def}{=} \mathcal{T}_f$ is a collection of subsets $U \subset A$ such that $A \setminus U$ is *either finite* or *all of* A ⟨ex. 3 p. 77 ∈ [1]⟩
- $\mathcal{T}_c$ **on** $A \overset{def}{=}$ all subsets $U \subset A$ such that $A \setminus U$ is *either countable* or *all of* A ⟨ex. 4 p. 77 ∈ [1]⟩
- ***Standard topology*** **on** $\mathbb{R} \overset{def}{=}$
  $\mathcal{T} =$ *generated by basis consisting of all open intervals* on $\mathbb{R}$,
  i.e. : *all* $]a, b[= \{x \mid a < x < b\}$. ⟨p. 81 ∈ [1]⟩
  This is just the *order topology* on $\mathbb{R}$, with order relation $<$. ⟨ex. 1 p. 85 ∈ [1]⟩
- ***Lower limit topology*** **on** $\mathbb{R} \overset{def}{=}$
  $\mathcal{T} =$ *generated by basis consisting of all intervals* of the form $[a, b[$.
  We call $\mathbb{R}$ with this topology $\mathbb{R}_l$. ⟨p. 82 ∈ [1]⟩
- ***K-topology*** **on** $\mathbb{R} \overset{def}{=}$ Let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$, then
  $\mathcal{T} =$ *generated by basis consisting of all sets* of the forms $]a, b[$ or $]a, b[\setminus K$.
  We call $\mathbb{R}$ with this topology $\mathbb{R}_K$. ⟨p. 82 ∈ [1]⟩
- ***Order topology*** $\overset{def}{=}$ *generated* by the *order topology basis* ⟨p. 84 ∈ [1]⟩
- ***Product topology*** **on** $A \times B \overset{def}{=}$
  the *topology generated* by the basis $\mathcal{B}$ where
  $\mathcal{B}$ is *all sets* $U \times V$ such that $U \underset{open}{\subset} A, V \underset{open}{\subset} B$.
  (the basis B is not a topology though) ⟨p. 86 ∈ [1]⟩
- ***Product topology*** **on** $\prod_{\alpha \in J} A_\alpha$, where A_α are *topological spaces* $\overset{def}{=}$
  the topology *generated* by the *product toplogy subbasis.* ⟨p. 114 ∈ [1]⟩
  We call $\prod_{\alpha \in J} A_\alpha$ with this topology a ***product space***.
  Can be described as: *all sets* $\prod_{\alpha \in J} U_\alpha$, where $(\forall \alpha \in J : U_\alpha \underset{open}{\subset} A_\alpha)$ and $U_\alpha = A_\alpha$ *except for finitely many* α ⟨thm. 19.1 p. 115 ∈ [1]⟩
- ***Box topology*** **on** $\prod_{\alpha \in J} A_\alpha \overset{def}{=}$
  The topology generated by the *box topology basis* ⟨p. 113, 114 ∈ [1]⟩
  Can be described as: *all sets* $\prod_{\alpha \in J} U_\alpha$, where $\forall \alpha \in J : U_\alpha \underset{open}{\subset} A_\alpha$ ⟨thm. 19.1 p. 115 ∈ [1]⟩
- ***Subspace topology*** $\mathcal{T}_B$ **on** $(A, \mathcal{T}) \overset{def}{=}$
  For $B \subset A : \mathcal{T}_B = \{B \cap U \mid U \in \mathcal{T}\}$.
  We have: $(B, \mathcal{T}_B) \underset{subspace}{\subset} (A, \mathcal{T})$ ⟨p. 88 ∈ [1]⟩
  Properties for $(B, \mathcal{T}_B) \underset{subspace}{\subset} (A, \mathcal{T})$:
  1) U ***open in*** B a.k.a. U ***open relative to*** $B \overset{def}{=} U \in \mathcal{T}_B$ ⟨p. 89 ∈ [1]⟩
  2) U ***open in*** A a.k.a. U ***open relative to*** $A \overset{def}{=} U \in \mathcal{T}$ ⟨p. 89 ∈ [1]⟩
  3) U *open in* B and B *open in* $A \Rightarrow U$ *open in* A ⟨lemma 16.2 p. 89 ∈ [1]⟩

- ***Ordered square*** I_0^2**, where** $I \underset{interval}{\subset} \mathbb{R} \overset{def}{=}$
  $I \times I$ with *dictionary order* and *order topology* ⟨ex. 3 p. 90 ∈ [1]⟩

2.2 Very Concrete Well-Known Topologies

- $\mathbb{R} \times \mathbb{R}$ with the *dictionary order*. Possible bases: ⟨ex. 2 p. 85 ∈ [1]⟩
  1) The collection of intervals $](a,b),(c,d)[$ where $a < c$ or $(a = c \wedge b < d)$
  2) The collection of intervals $](a,b),(c,d)[$ where $(a = c \wedge b < d)$

- $\mathbb{Z}_+$ with the *order topology* is the *discrete topology.* ⟨ex. 3 p. 85 ∈ [1]⟩

- The *order toplogy* for $\{1,2\} \times \mathbb{Z}_+$ with the *dictionary order* is
  not the *discrete topology* ⟨ex. 4 p. 85 ∈ [1]⟩

2.3 Common Well-Known Topology Bases

Premises: A, B ***sets***

- The collection of *all circular regions* (*interiors* of circles) in $\mathbb{R}^2$ ⟨ex. 1 p. 78 ∈ [1]⟩

- The collection of *all rectangular regions* (*interiors* of rectangles) in $\mathbb{R}^2$ ⟨ex. 2 p. 79 ∈ [1]⟩
  This is a basis for the *standard topology* on $\mathbb{R}^2$
  (the *product* of 2 *order topologies*: $\mathbb{R} \times \mathbb{R}$) ⟨ex. 1 p. 87 ∈ [1]⟩

- The collection of *all one-point subsets* of *any set* A.
  This is the basis for the *discrete topology* ⟨ex. 3 p. 79 ∈ [1]⟩

- *Topologies* which *also qualify* as *bases*:
  standard topology, *K-topology*, *lower limit topology.* ⟨p. 82 ∈ [1]⟩

- **Basis for the *order topology* on A, where A is an *ordered set* with *simple order relation*** $< \overset{def}{=} \mathcal{B}$ is given by:
  1) *All open intervals* $]a.b[$ in A
  2) If A has a *smallest element* a_0: Then also *all intervals* $[a_0, b[$
  3) If A has a *largest element* b_0: Then also *all intervals* $]a, b_0]$
  ⟨p. 84 ∈ [1]⟩

- **Alternative basis $\mathcal{B}$ for the *product topology* on** $A \times B$:
  If $\mathcal{B}$ is *basis* for the *topology* on A and $\mathcal{C}$ *basis* for the *topology* on B, then $\mathcal{D} = \{(B,C) \mid B \in \mathcal{B} \ \wedge \ C \in \mathcal{C}\}$ is *basis* for the *topology* on $A \times B$ ⟨thm. 15.1 p. 86 ∈ [1]⟩

- **Alternative basis $\mathcal{B}$ for the *product topology* on** $\prod_{\alpha \in J} A_\alpha$,
  where each A_α is *given by the basis* $\mathcal{B}_\alpha \overset{def}{=}$
  $\mathcal{B} =$ *all sets* of the *form* $\prod_{\alpha \in J} B_\alpha$, where for $\forall I \underset{finite}{\subset} J$ we have that:
  $\forall \alpha \in I : B_\alpha \in \mathcal{B}_\alpha$ and $\forall \alpha \in J \backslash I : B_\alpha = A_\alpha$ ⟨thm. 19.2 p. 116 ∈ [1]⟩
  I.e. $B_\alpha \in \mathcal{B}_\alpha$ for *finitely many* α and $B_\alpha = A_\alpha$ for the *other* α.

- **Basis $\mathcal{B}$ for the *box topology* on** $\prod_{\alpha \in J} A_\alpha \overset{def}{=}$
  Let $\{A_\alpha\}_{\alpha \in J}$ be an *indexed family* of *topological spaces*, the basis $\mathcal{B}$ on the *product space* $\prod_{\alpha \in J} A_\alpha$ is given by:
  $\mathcal{B} =$ *all sets* of the *form* $\prod_{\alpha \in J} U_\alpha$, where $\forall \alpha \in J : U_\alpha \underset{open}{\subset} A_\alpha$ ⟨p. 114 ∈ [1]⟩

- **Alternative basis $\mathcal{B}$ for the *box topology* on $\prod_{\alpha \in J} A_\alpha$,**
  where each A_α is *given by the basis* $\mathcal{B}_\alpha$ $\overset{def}{=}$
  $\mathcal{B} =$ *all sets* of the *form* $\prod_{\alpha \in J} B_\alpha$, where $\forall \alpha \in J : B_\alpha \in \mathcal{B}_\alpha$ ⟨thm. 19.2 p. 116 $\in$ [1]⟩

- **A basis $\mathcal{B}_B$ for the *subspace topology* on $(A, \mathcal{T})$:**
  For $B \subset A$, and $\mathcal{B}$ a *basis* for $(A, \mathcal{T})$, then
  $\mathcal{B}_B = \{B \cap Y \mid B \in \mathcal{B}\}$. ⟨lemma 16.1 p. 89 $\in$ [1]⟩

2.4 Common Well-Known Topology Subbases

Premises: A, B ***sets***

- **Subbasis for the *order topology* on A** $\overset{def}{=}$
  The *open rays* on A: $]a, +\infty[= \{x \mid x > a\}$ and $] - \infty, a[= \{x \mid x < a\}$ ⟨p. 86 $\in$ [1]⟩

- **Subbasis for the *product topology* on $A \times B$** $\overset{def}{=}$
  $\mathcal{S} = \{\pi_1^{-1}(U) \mid U \textit{ open in } A\} \cup \{\pi_2^{-1}(V) \mid V \textit{ open in } B\}$, where
  $\pi_1 : A \times B \to A, \pi_2 : A \times B \to B$ are the *projections* ⟨thm. 15.2 p. 88 $\in$ [1]⟩

- **Subbasis $\mathcal{S}$ for the *product topology* on $\prod_{\alpha \in J} A_\alpha$** $\overset{def}{=}$
  Let $\mathcal{S}_\alpha = \{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \underset{open}{\subset} A_\alpha\}$, then
  $\mathcal{S} = \cup_{\alpha \in J} \mathcal{S}_\alpha$, where $\pi_\alpha : (\prod_{\alpha \in J} A_\alpha) \to A_\alpha$ are the *projections* ⟨p. 114 $\in$ [1]⟩

2.5 Theorems Abount Well-Known Topolgies

- Let $B_\alpha \underset{subspace}{\subset} A_\alpha$ for all $\alpha \in J$. Then $(\prod B_\alpha) \underset{subspace}{\subset} (\prod A_\alpha)$ if:
  either *both* have the *product topology* or *both* have the *box topology*. ⟨thm. 19.3 p. 116 $\in$ [1]⟩

3 Comparison of Topologies

A, B ***sets***, $\mathcal{T}, \mathcal{T}'$ ***topologies* on** A

- For $\mathcal{T} \subset \mathcal{T}'$ we *define*:
  $\mathcal{T}$ ***coarser*** / ***smaller* than** $\mathcal{T}'$
  $\mathcal{T}'$ ***finer*** / ***larger* than** $\mathcal{T}$
  Beware of the terms *weaker* / *stronger*! (some people have incompatiple meanings for these terms)
  ⟨p. 77 ∈ [1]⟩

- For $\mathcal{T} \subsetneq \mathcal{T}'$ we *define*:
  $\mathcal{T}$ ***strictly coarser*** / ***strictly smaller* than** $\mathcal{T}'$
  $\mathcal{T}'$ ***strictly finer*** / ***strictly larger* than** $\mathcal{T}$
  ⟨p. 77 ∈ [1]⟩

- $\mathcal{T}$ ***comparable* with** $\mathcal{T}'$ $\stackrel{def}{=}$ either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$ ⟨p. 77 ∈ [1]⟩

- Let $\mathcal{B}$ and $\mathcal{B}'$ be *bases* for the *topologies* $\mathcal{T}$ and $\mathcal{T}'$, repectively, on A.
  Then the following are *equivalent*:
  1) $\mathcal{T}'$ is *finer* than $\mathcal{T}$
  2) $\forall a \in A : \forall B \in \mathcal{B}, a \in B : \exists B' \in \mathcal{B}' : a \in B' \subset B$
  ⟨lemma 13.3 p. 81 ∈ [1]⟩

3.1 Comparison of Well-Known Topologies

Premises: A, B ***sets***

- If A *subspace* of X and B *subspace* of Y, then
  the *product topology* on $A \times B$ is the *same as*
  the topology $A \times B$ *inherits* as a *subspace* of $X \times Y$. ⟨lemma 16.3 p. 89 ∈ [1]⟩

- **Warning**: Let A be an *ordered set* with *order topology*, and let $B \subset A$.
  The *order relation* on A *restricted to* B makes B an *ordered set*. However:
  The *resuling order topology* on B $\underset{in\ general}{\neq}$ the *topology* that B *inherits*
  as a *subspace* of A. ⟨p. 90 ∈ [1]⟩
  Examples:
  1) The *topology of* $[0,1]$ as a *subspace* of $\mathbb{R}$ = *order topology* on $[0,1]$ ⟨ex. 1 p. 90 ∈ [1]⟩
  2) The *topology of* $B = [0,1[\cup \{2\}$ as a *subspace* of $\mathbb{R} \neq$ *order topology* on B ⟨ex. 2 p. 90 ∈ [1]⟩
  3) The *topology of* $B = [0,1] \times [0,1]$ with *dictionary order* as a *subspace* of $\mathbb{R} \times \mathbb{R} \neq$ *order topology* on $[0,1] \times [0,1]$ ⟨ex. 3 p. 90 ∈ [1]⟩

  To avoid ambiguity: When A is an *ordered set* in *order topology* and
  $B \subset A$, we *assume* that B is *given* the *subspace topology* (unless otherwise
  stated). ⟨p. 91 ∈ [1]⟩

- If A is an *ordered set* with *order topology* and $B \underset{convex\ subset}{\subset} B$, then
  order topology on B = topology B *inherited* as a *subspace* of A. ⟨thm. 16.4 p. 91 ∈ [1]⟩

- *standard topology* $\subset$ *lower limit topology* ⟨lemma 13.4 p. 82 ∈ [1]⟩

- *standard topology* $\subset$ *K-topology* ⟨lemma 13.4 p. 82 ∈ [1]⟩

- *lower limit topology* is *not comparable* to *K-topology* ⟨lemma 13.4 p. 82 ∈ [1]⟩

- *Product topology* on $\mathbb{R}^J \subset$ *uniform topology* on $\mathbb{R}^J \subset$ *box topology* on $\mathbb{R}^J$.
  If J is *infinite*, they are *all different*. ⟨thm. 20.4 p. 124 ∈ [1]⟩

4 Topological Properties

- ***Topological property*** $\overset{def}{=}$ property expressed *only via open sets* ⟨p. 105 ∈ [1]⟩

4.1 Closed Sets

Premises: $(A, \mathcal{T})$ ***topological space***

- **A *subset* U of A is *closed*** $\overset{def}{=} U \setminus A$ *open.* ⟨p. 93 ∈ [1]⟩
- The following hold:
  1) $\emptyset$ and A are *closed*
  2) *Arbitrary intersections* of *closed sets* are *closed*
  3) *Finite unions* of *closed sets* are *closed*
- For $(B, \mathcal{T}_B) \underset{subspace}{\subset} (A, \mathcal{T})$ then the following hold:
  1) U *closed* in $B \Leftrightarrow \exists V \underset{closed}{\subset} A : U = B \cap V$ ⟨thm. 17.2 p. 94 ∈ [1]⟩
  2) U *closed* in B and B *closed* in $A \Rightarrow U$ *closed in* A ⟨thm. 17.3 p. 95 ∈ [1]⟩

4.2 Closure, Interior

Premises: $(A, \mathcal{T})$ ***topological space***

- ***Interior* of** $B \subset A \equiv \text{Int } B \overset{def}{=} \bigcup_{B_i \underset{open}{\subset} A, B_i \subset B} B_i$ ⟨p. 95 ∈ [1]⟩
- ***Closure* of** $B \subset A \equiv \text{Cl } B$ w.r.t. $A \equiv \text{Cl}_A B \equiv \overline{B} \overset{def}{=}$ $\text{Cl}_A B = \bigcap_{B \subset B_i \underset{closed}{\subset} A} B_i$ ⟨p. 95 ∈ [1]⟩
- $\text{Int } A \subset A \subset \overline{A}$ ⟨p. 95 ∈ [1]⟩
- A *open* $\Rightarrow A = \text{Int } A$ ⟨p. 95 ∈ [1]⟩
- A *closed* $\Rightarrow A = \overline{A}$ ⟨p. 95 ∈ [1]⟩
- For $(B, \mathcal{T}_B) \underset{subspace}{\subset} (A, \mathcal{T})$ and $U \subset B$, we have:
  1) **Warning**: $\text{Cl}_B U \underset{in\ general}{\neq} \text{Cl}_A U$. In such cases *we assume* $\text{Cl } U = \text{Cl}_A U$ ⟨p. 95 ∈ [1]⟩
  2) $\text{Cl}_B U = (\text{Cl}_A U) \cap B$ ⟨thm. 17.4 p. 95 ∈ [1]⟩
- For $U \subset A$, we have:
  $x \in \text{Cl}_A U \Leftrightarrow \forall V \in \mathcal{T}, x \in V : U \cap A \neq \emptyset$
  (i.e. *every neighborhood* of x *intersects* A) ⟨thm. 17.5 p. 96 ∈ [1]⟩
- If $\mathcal{T}$ is given by a basis $\mathcal{B}$, then:
  $x \in \text{Cl}_A A \Leftrightarrow \forall B \in \mathcal{B}, x \in B : B \cap A \neq \emptyset$ ⟨thm. 17.5 p. 96 ∈ [1]⟩

4.2.1 Examples

- In the *finite complement topology* on A, the *closed sets* are:
  The set A itself and *all finite subsets* of A. ⟨ex. 3 p. 93 ∈ [1]⟩
- In the *discrete topology* on A:
  All sets are open, so *all sets are closed as well.* ⟨ex. 4 p. 93 ∈ [1]⟩
- Consider the *subspace* $A =]0, 1]$ of $\mathbb{R}$. The set $U =]0, \frac{1}{2}[$ is a *subset* of A.
  $\text{Cl}_{\mathbb{R}} U = [0, \frac{1}{2}] \neq \text{Cl}_A U = [0, \frac{1}{2}] \cap A =]0, \frac{1}{2}]$ ⟨ex. 7 p. 97 ∈ [1]⟩

4.3 Limit Points

Premises: $(A, \mathcal{T})$ ***topological space***

- ***Limit point / cluster point / point of accumulation*** x **of** $A \overset{def}{=}$ Every *neighborhood* of x *intersects* A in *some other* point $y \neq x \equiv$ $x \in (\mathrm{Cl}_A(A \setminus \{x\}))$ ⟨p. 97 ∈ [1]⟩
- **Notation:** When x_n is *sequence of points* in A *converging* to $x \in A$, we write $x_n \to x$, x is the *limit* of x_n. ⟨p. 100 ∈ [1]⟩
- For a subset $U \subset A$:
  1. Let B' be the set of *all limit points* of B, then $\mathrm{Cl}_A B = B \cup B'$ ⟨thm. 17.6 p. 97 ∈ [1]⟩
  2. B *closed* $\Leftrightarrow$ (*all limit points of* B) $\underset{subset}{\subset} B$ ⟨cor. 17.7 p. 98 ∈ [1]⟩
- For $\{A_\alpha\}$ an *indexed family of spaces*, let $B_\alpha \subset A_\alpha$ for each α. If $\prod A_\alpha$ has either *product topology* or *box topology*, then $\prod \overline{B_\alpha} = \overline{\prod B_\alpha}$. ⟨thm. 19.5 p. 116 ∈ [1]⟩

4.3.1 Hausdorff Spaces

Premises: $(A, \mathcal{T})$ ***Hausdorff space***

- $(A, \mathcal{T})$ ***Hausdorff space*** $\overset{def}{=}$ $\forall x, y \in A, x \neq y$: there *exists disjoint neighborhoods* U_1 and U_2 of x and y respectively. ⟨p. 98 ∈ [1]⟩
- Every *finite point set* in any *Hausdorff space* is *closed.* ⟨thm. 17.8 p. 99 ∈ [1]⟩
- A *sequence of points* of A *converges* to *at most one point* of A. ⟨thm. 17.10 p. 99 ∈ [1]⟩
- $(A, \mathcal{T})$ *Hausdorff* $\Leftrightarrow$ the *diagonal* $\Delta = \{(a, a) \mid a \in A\}$ is *closed* in $A \times A$. ⟨exc. 17; 13 p. 101 ∈ [1]⟩
- Well-known Hausdorff spaces:
  - The *order topology* of *any simply ordered set* ⟨exc. 17;10 p. 101 ∈ [1]⟩
  - A *subspace* of a *Hausdorff space* ⟨exc. 17;12 p. 101 ∈ [1]⟩
- If $\forall \alpha \in J : A_\alpha$ *Hausdorff*, then $\prod A_\alpha$ is *Hausdorff* in *both* the *product topology* and the *box topology.* ⟨thm. 19.4 p. 116 ∈ [1]⟩

4.3.2 The T_1 axiom

Premises: $(A, \mathcal{T})$ ***topological space satisfying the*** T_1 ***axiom***

- ***The*** T_1 ***axiom*** $\overset{def}{=}$ *finite point sets* are *closed* (weaker than Hausdorff property) ⟨p. 99 ∈ [1]⟩
- Let $B \underset{subset}{\subset} A$. Then the *point* x is a *limit point of* $B \Leftrightarrow$ *every neighborhood* of x *contains infinitely many points* of B ⟨thm. 17.9 p. 99 ∈ [1]⟩

5 Continuous Functions

5.1 Well-Known Functions

Premises: A, B ***topological spaces***

- $f : A \to B$ ***open map*** $\overset{def}{=} U \underset{open}{\subset} A \Rightarrow f(U) \underset{open}{\subset} B$ ⟨exc. 4 p.92 ∈ [1]⟩
- If $U \underset{open}{\subset} X_i$ then,
$\pi_i^{-1}(U) = (X_1 \times X_2 \times \cdots \times U \times \cdots \times X_n) \underset{open}{\subset} (X_1 \times X_2 \times \cdots \times X_i \times \cdots \times X_n)$
⟨p. 87 ∈ [1]⟩

5.2 Continuous Functions

Premises: A, B ***topological spaces***

- $f : A \to B$ ***continuous*** (*relative* to the *topologies* on A and B) $\overset{def}{=}$
$\forall V \underset{open}{\subset} B : f^{-1}(V) \underset{open}{\subset} A$. ⟨p. 102 ∈ [1]⟩
So *continuity depends both* on *function* and *topologies* for A and B.
Example (lower limit topology and identity function):
$id : \mathbb{R} \to \mathbb{R}_l$ is *not continuous*, but $id : \mathbb{R}_l \to \mathbb{R}$ *is continuous.*
- $f : A \to B$ ***continuous at*** $a \in A \overset{def}{=}$
$\forall V$ *neighborhood* of $f(a) : \exists U$ *neighborhood* of $a : f(U) \subset V$ ⟨p. 104 ∈ [1]⟩
- If *topology* of B is *given by the topology* BASIS $\mathcal{B}$, then:
$(\forall B_i \in \mathcal{B} : f^{-1}(B_i) \underset{open}{\subset} A) \Rightarrow f : A \to B$ *continuous* ⟨p. 103 ∈ [1]⟩
Prooving: Any open set V (a *union of basis elements*): $V = \bigcup_{\alpha \in J} B_\alpha$. So $f^{-1}(V) = \bigcup_{\alpha \in J}(f^{-1}(B_\alpha))$
- If *topology* of B is *given by the topology* SUBBASIS $\mathcal{S}$, then:
$(\forall S_i \in \mathcal{S} : f^{-1}(S_i) \underset{open}{\subset} A) \Rightarrow f : A \to B$ *continuous* ⟨p. 103 ∈ [1]⟩
- For *top. spaces* $(A, \mathcal{T})$, $(B, \mathcal{T}')$ and $f : A \to B$, the following are *equiv.*:
    1. f *continuous* ⟨thm. 18.1 p. 104 ∈ [1]⟩
    2. $\forall U \subset A : f(\overline{U}) = \overline{f(U)}$ ⟨thm. 18.1 p. 104 ∈ [1]⟩
    3. $\forall V \underset{closed}{\subset} B : f^{-1}(V) \underset{closed}{\subset} A$ ⟨thm. 18.1 p. 104 ∈ [1]⟩
    4. $\forall a \in A : \forall V$ *neighborhood* of $f(a) : \exists U$ *neighborhood* of $a : f(U) \subset V$
    ⟨thm. 18.1 . 104 ∈ [1]⟩
- Let $f : A \to (\prod_{\alpha \in J} A_\alpha)$ be given by $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to A_\alpha$ for each α. Let $\prod A_\alpha$ have the *product topology.* Then:
f *continuous* $\Leftrightarrow \forall \alpha \in J : f_\alpha$ *continuous* ⟨thm. 19.6 p. 117 ∈ [1]⟩
Warning: Does *not* hold for *box topology*!

5.3 Constructing Continuous Functions

Premises: $(A, \mathcal{T}), (B, \mathcal{T}'), (C, \mathcal{T}'')$ ***topological spaces***

- If one of the following holds, then $f : A \to B$ is *continuous*:

  1. *Constant function*: If $f : A \to B$ maps *all* of A into a *single point* $y_0 \in B$ ⟨thm. 18.2 p. 108 ∈ [1]⟩
  2. *Inclusion*: If B is a *subspace* of A and the *inclusion function* $f : B \to A$ is *continuous* ⟨thm. 18.2 p. 108 ∈ [1]⟩
  3. *Composites*: $g : A \to C$ *continuous*, $h : C \to B$ *continuous* and $f = h \circ g$ ⟨thm. 18.2 p. 108 ∈ [1]⟩
  4. *Restricting the domain*: If $g : C \to B$ *continuous* and A *subspace* of C and $f = g|_A$ ⟨thm. 18.2 p. 108 ∈ [1]⟩
  5. *Restricting range*: If $g : A \to C$ *continuous* and B is a *subspace* of C *containing* $g(A)$ and f is g with *restricted range* ⟨thm. 18.2 p. 108 ∈ [1]⟩
  6. *Expanding range*: If $g : A \to C$ *continuous* and C is a *subspace* of B f is g with *expanded range* ⟨thm. 18.2 p. 108 ∈ [1]⟩
  7. *Local continuity*: If A *can be written* as the *union* of *open sets* U_α, such that $\forall \alpha : f|_{U_\alpha}$ is *continuous* ⟨thm. 18.2 p. 108 ∈ [1]⟩
  8. *Pasting lemma*: Let $A = S \cup T$ and $S \cap T \neq \emptyset$, for $S \underset{closed}{\subset} A, T \underset{closed}{\subset} A$ (or $S \underset{open}{\subset} A, T \underset{open}{\subset} A$). If $g : S \to B$ *continuous*, $h : T \to B$ *continuous* and $\forall x \in S \cup T : g(x) = h(x)$ and $f(x) = \begin{cases} g(x), & x \in S \\ h(x), & x \in T \end{cases}$
     ⟨thm. 18.3 p.108 ∈ [1]⟩

- Let $f : A \to B \times C$ be given by $f(a) = (f_1(a), f_2(a))$, then
  f *continuous* $\Leftrightarrow$ f_1, f_2 *continuous* ⟨thm. 18.4 p. 110 ∈ [1]⟩

5.4 Homeomorphisms

Premises: $(A, \mathcal{T}), (B, \mathcal{T}')$ ***topological spaces***

- $f : A \to B$ ***homeomorphism*** $\overset{def}{=}$ f *bijective* and f and f^{-1} *both continuous* ⟨p. 105 ∈ [1]⟩

- $f : A \to B$ ***homeomorphism*** $\equiv$ f *bijective*, $f(U) \underset{open}{\subset} B \Leftrightarrow U \underset{open}{\subset} A$ ⟨p. 105 ∈ [1]⟩

- *Homeomorphisms preserve topological properties* ⟨p. 105 ∈ [1]⟩
  (*Homeomorphisms* is the *topology's equivalent* of *isomorphisms* in *algebra*)

- $f : A \to B$ **is a *topological imbedding*** $\overset{def}{=}$
  $f : A \to B$ *injective*, *continuous* and $f : A \to C$ *homeomorphism*,
  where $C = f(A)$ is considered a *subspace* of B ⟨p. 105 ∈ [1]⟩

- Example: *Order preserving* and *bijective* $\Rightarrow$
  homeomorphic in the *order topology* ⟨p. 106 ∈ [1]⟩

6 Metric Topology

Premises: A, B ***topological spaces***, (M, d) ***metric space***, d ***metric* of** M

- ***Basis* for the *metric topology*** $\overset{def}{=}$
  Basis $\mathcal{B}$ = all *open spheres* $K_d(x, r)$ for $x \in A, r > 0$, where
  $K_d(x, r) = \{y \mid d(x, y) < r\}$ ⟨p. 119 ∈ [1]⟩
  From this we can get the usual definition of "open" using open spheres.
  Prooving: Use sphere lemma

- For *topological space* A, A **is *metrizable*** $\overset{def}{=}$
  There *exists* a *metric* d on A that *induces* the *topology* of A.
  A ***metric space*** is a *metrizable space* A together with a *specific metric* d that gives the *topology* of A. ⟨p. 120 ∈ [1]⟩

- **A set $U \subset M$ is *bounded*** $\overset{def}{=}$
  $\exists m \in \mathbb{R} : \forall x_1, x_2 \in U : d(x_1, x_2) < m.$ ⟨p. 121 ∈ [1]⟩
  Boundedness is *not* a *topological property*, since it depends on d.

- For $U \subset M$, U *bounded* and $U \neq \emptyset$, the *diameter* of U $\overset{def}{=}$
  diam $U = \sup\{d(x_1, x_2) \mid x_1, x_2 \in U\}$ ⟨p. 121 ∈ [1]⟩

- ***Standard bounded metric*** $\bar{d} : M \times M \to \mathbb{R}$ **corresponding to** d $\overset{def}{=}$
  $\bar{d}(x, y) = \min\{d(x, y), 1\}$.
  $\bar{d}$ is a *metric* that *induces* the *same topology* as d. ⟨thm. 20.1 p. 121 ∈ [1]⟩

- *Misc. definitions* for $\vec{x} = (x_1, \dots, x_2) \in \mathbb{R}$: ⟨p. 121 ∈ [1]⟩
  - **The *norm* of $\vec{x}$:** $\|\vec{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$
  - **The *euclidean metric* d on $\mathbb{R}^n$:** $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| = \sqrt{(\vec{x}_1 - \vec{y}_1)^2 + \cdots + (\vec{x}_n - \vec{y}_n)^2}$
  - **The *square metric* ρ on $\mathbb{R}^n$:** $\rho(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$

- Let d and d' be *two metrics* on A, and let $\mathcal{T}$ and $\mathcal{T}'$ be the *topologies* they *induce*.
  T' *finer than* $\mathcal{T}$ (i.e. $\mathcal{T} \subset \mathcal{T}'$) $\Leftrightarrow$
  $\forall x \in A : \forall \epsilon > 0 : \exists \delta > 0 : K_{d'}(x, \delta) \subset K_d(x, \epsilon)$ ⟨lemma 20.2 p. 122 ∈ [1]⟩

- The *topologies* on $\mathbb{R}^n$ *induced* by the *euclidean metric* d and the *square metric* ρ are the *same* as the *product topology* on $\mathbb{R}^n$. ⟨thm. 20.3 p. 123 ∈ [1]⟩

- In $\mathbb{R}^\omega$, $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$ and $\rho(\vec{x}, \vec{y}) = \sup\{|x_n - y_n|\}$ does *not always converge*, and does thus *not always* define a *metric* of $\mathbb{R}^\omega$. ⟨p. 124 ∈ [1]⟩

- **The *uniform metric* $\bar{\rho}$ on $\mathbb{R}^J$** for *arbitrary index set* J $\overset{def}{=}$
  $\bar{\rho}(\vec{x}, \vec{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$, where $\bar{d}$ is *standard bounded metric* on $\mathbb{R}^J$ and $\vec{x} = (x_\alpha)_{\alpha \in J}$, $\vec{y} = (y_\alpha)_{\alpha \in J}$. ⟨p. 124 ∈ [1]⟩

- **The *uniform topology* on $\mathbb{R}^J$** for *arbitrary index set* J $\overset{def}{=}$
  The *topology induced* by the *uniform metric* on $\mathbb{R}^J$. ⟨p. 124 ∈ [1]⟩

- **We define** $D(\vec{x}, \vec{y}) = \sup\{\frac{\bar{d}(x_i, y_i)}{i}\}$ **on** $\mathbb{R}^\omega$ **for** $\vec{x}, \vec{y} \in \mathbb{R}^\omega$,
  where $\bar{d}$ is the *standard bounded metric*.
  D *induces* the *product topology* on $\mathbb{R}^\omega$ ⟨thm. 20.5 p. 125 ∈ [1]⟩

- *Metrics* vs. *topologies*: ⟨p. 129 ∈ [1]⟩
  - $B \underset{subspace}{\subset} A$, d is a *metric* for the *topology on* $A \Rightarrow$ $d|_{B\times B}$ is a *metric* for the *topology on* B
  - *Some order topologies* are *metrizable* (e.g. $\mathbb{Z}_+, \mathbb{R}$) and *others are not*
  - *Hausdorff axiom holds* for *every metric topology*
  - *Countable products* of *metrizable spaces* are *metrizable*

- Let $f : A \to B$ and A, B be *metrizable topological spaces* with *metrics* d_A, d_B respectively, then: f *continuous* $\Leftrightarrow$ $\forall x \in A : \forall \epsilon > 0 : \exists \delta > 0 : (d_A(x,y) < \delta) \Rightarrow (d_B(f(x), f(y)) < \epsilon)$ ⟨thm. 21.1 p. 129 ∈ [1]⟩

- Let $U \subset A$. There is a *sequence of points* of U converging to $x \Rightarrow x \in \overline{U}$. The *converse* hold if A is *metrizable* (or just satisfies the *first countability axiom*). Called the *sequence lemma.* ⟨lemma 21.2 p. 130 ∈ [1]⟩

- Let $f : A \to B$. If f is *continuous*, then for *every convergent sequence* $x_n \to x$, $f(x_n)$ *converges* to $f(x)$.
  The *converse* hold if A is *metrizable* (or just satisfies the *first countability axiom*). ⟨lemma 21.3 p. 130 ∈ [1]⟩

- The *topological space* A has a *countable basis* at $x \in A \overset{def}{=}$
  There *exists* a *countable collection* of *neighborhoods* of x, such that *any neighborhood* U of x *contains at least one* of the U_n. ⟨p. 130 ∈ [1]⟩

- The *topological space* A satisfies the ***first countability axiom*** $\overset{def}{=}$
  $\forall x \in A : x$ has a *countable basis* ⟨p. 131 ∈ [1]⟩

- Let U be a *set* and $f_n : U \to M$ a *sequence of functions*, then:
  f_n ***converges uniformly* to** $f : U \to M \overset{def}{=}$
  $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n > N : \forall x \in U : d(f_n(x), f(x)) < \epsilon$ ⟨p. 131 ∈ [1]⟩

- *Uniform limit theorem*: Let U be a *set* and $f_n : U \to M$ a *sequence of continuous functions.*
  If (f_n) *converges uniformly* to f, then f is *continuous* ⟨thm. 21.6 p. 132 ∈ [1]⟩

- In the space $\mathbb{R}^X$ of *functions* $f : X \to \mathbb{R}$ with the *uniform metric* $\bar{\rho}$:
  A *sequence* (f_n) *converges uniformly* to $f \Leftrightarrow f_n$ *converges* to f w.r.t. $\bar{\rho}$. ⟨p. 132 ∈ [1]⟩

6.1 Well-Known Facts

- The *discrete topology* is *induced* as a *metric topology* by the *discrete metric*. ⟨ex. 1 p. 120 ∈ [1]⟩
- The *order topology* on $\mathbb{R}$ is *induced* as a *metric topology* by the *standard metric* on $\mathbb{R} : d(x, y) = |x - y|$. ⟨ex. 2 p. 120 ∈ [1]⟩
- *Non-metrizable spaces*:
  - $\mathbb{R}^J$, where J is *uncountable* with *product topology*, *box topology* or *uniform topology* ⟨p. 125 ∈ [1]⟩
  - $\mathbb{R}^J$, where J is *countable* with *box topology* or *uniform topology* ⟨p. 125 ∈ [1]⟩
- *Continuous functions*, where f, g are *continuous*:
  - $(f+g), (f-g)$ and $(f \cdot g)$ are *continuous*
  - If $\forall x : g(x) \neq 0$, then (f/g) is *continuous*
  - On $\mathbb{R} \times \mathbb{R} : +, -, \cdot$ *operations* ⟨lemma 21.4 p. 131 ∈ [1]⟩
  - On $\mathbb{R} \times (\mathbb{R} \backslash \{0\}) : /$, the *divide operation* ⟨lemma 21.4 p. 131 ∈ [1]⟩

7 Quotient Topology

Premises: A, B ***topological spaces***

- Let $p : A \to B$ be *surjective.*
  p **is a *quotient map* / has *strong continuity*** $\stackrel{def}{=}$
  $\forall U \subset B : (U \textit{ open} \Leftrightarrow p^{-1}(U) \textit{ open}) \equiv$
  $\forall U \subset B : (U \textit{ closed} \Leftrightarrow p^{-1}(U) \textit{ closed})$ ⟨p. 137 ∈ [1]⟩

- $U \subset A$ **is a *saturated subset* of** A w.r.t. *surjective map* $p : A \to B \stackrel{def}{=}$
  $\forall y \in A : (p^{-1}(\{y\}) \textit{ intersects } U \Rightarrow p^{-1}(\{y\}) \subset U)$ ⟨p. 137 ∈ [1]⟩
  Thus U is *saturated* if it *equals* the *complete inverse image* of a *subset* of B.

- Let $p : A \to B$ be *surjective.* $p : A \to B$ is a *quotient map* $\equiv$ ⟨p. 137 ∈ [1]⟩
  p *continuous* and maps *saturated open sets* of A *into open sets* of $B \equiv$
  p *continuous* and maps *saturated closed sets* of A *into closed sets* of B

- ***Open quotient map*** $p : A \to B \stackrel{def}{=}$ ⟨p. 137 ∈ [1]⟩
  p *quotient map* and p *open map* (I.e.: $\forall U \underset{open}{\subset} A : p(U)$ *open* in B)

- ***Closed quotient map*** $p : A \to B \stackrel{def}{=}$ ⟨p. 137 ∈ [1]⟩
  p *quotient map* and p *closed map* (I.e.: $\forall U \underset{closed}{\subset} A : p(U)$ *closed* in B)

- $p : A \to B$ *continuous, surjective* and *either open* or *closed* $\Rightarrow$
  p *quotient map.* ⟨p. 137 ∈ [1]⟩
  There exists quotient maps that are *neither open nor closed.*

- Let X be a *set* and $p : A \to X$ a *surjective map.*
  The *quotient topology* $\mathcal{T}$ (which is *unique*!) **on X *induced* by** $p \stackrel{def}{=}$
  The *topology* $\mathcal{T}$ on X relative to which p is a *quotient map.* ⟨p. 138 ∈ [1]⟩
  $\mathcal{T}$ is *defined* as the *subsets* U of X, such that $p^{-1}(U)$ are *open* in A.

- Let $A*$ be a *partition* of A into *disjoint subsets* whose union is A.
  Let $p : A \to A*$ be the *surjective map* that *carries each point* of A to the *element* of $A*$ that *contains* it. In the *quotient topology induced* by p, $A*$ is called the ***quotient space* of** A. ⟨p. 179 ∈ [1]⟩
  Often called an ***identification space* of** A or a ***decomposition space* of** A.

7.1 Quotient maps vs. quotient space

Premises: A, B ***topological spaces***

- Warning: If $p : A \to B$ is a *quotient map* and $C \underset{subspace}{\subset} A$, then $q : C \to p(C)$ obtained by *restricting p need not* be a *quotient map*! ⟨p. 140 ∈ [1]⟩
- Let $p : A \to B$ be a *quotient map* and $C \underset{subspace}{\subset} A$, where C is *saturated* w.r.t. p. Let $q : C \to p(C)$ be p *restricted* to C,
  then q is a *quotient map* if *one of the following holds*:
  1. If C is *either open* or *closed*. ⟨thm. 22.1 p. 140 ∈ [1]⟩
  2. If p is *either* an *open map* or a *closed map*. ⟨thm. 22.1 p. 140 ∈ [1]⟩
- p, q *quotient maps* $\Rightarrow p \circ q$ *quotient map* ⟨p. 141 ∈ [1]⟩
- If p, q are *quotient maps*, then (p, q) is a *quotient map* if *one of the following holds* (but *not* in *general*!):
  1. If the *spaces* have *local compactness* ⟨p. 141 ∈ [1]⟩
  2. If p and q are *open maps* ⟨p. 141 ∈ [1]⟩
- If A is a *Hausdorff space*, then:
  1. If *each element* of $A*$ is a *closed subset* of A, then the *quotient space* $A*$ satisfies the T_1-*axiom* ⟨p. 141 ∈ [1]⟩
  2. For $A*$ to be *Hausdorff*, it's *harder* to find conditions ⟨p. 141 ∈ [1]⟩
- Let A, B, C be *topological spaces* and $p : A \to B$ be a *quotient map*.
  Let $g : A \to C$ be a map which is *constant* on *each set* $p^{-1}(\{b\})$, for $b \in B$.
  Then g *induces* the map $f : B \to C$, such that $f \circ p = g$, and we have:
  1. f *continuous* $\Leftrightarrow g$ *continuous* ⟨thm. 22.2 p. 142 ∈ [1]⟩
  2. f *quotient* $\Leftrightarrow g$ *quotient* ⟨thm. 22.2 p. 142 ∈ [1]⟩
- Let $g : A \to B$ be a *surjective continuous map*. Let $A*$ be the *following collection* of *subsets* of A: $A* = \{g^{-1}(\{b\}) \mid b \in B\}$.
  Give $A*$ the *quotient topology*, then we have:
  1. g *induces* a *bijective continuous map* $f : A* \to B$, which is a *homeomorphism* if and only if g is a *quotient map*. ⟨cor. 22.3 p. 142 ∈ [1]⟩
  2. B *Hausdorff* $\Rightarrow A*$ *Hausdorff* ⟨cor. 22.3 p. 142 ∈ [1]⟩
- Warning: The *product* of 2 *quotient maps* need *not* be a *quotient map* ⟨ex. 7 p. 143 ∈ [1]⟩
- **A *retraction* of A into U, where $U \subset A$** $\overset{def}{=}$
  a *continuous map* $r : A \to U$, where $\forall a \in U : r(a) = a$ ⟨exc. 22;2 p. 144 ∈ [1]⟩
- f is a *retraction* $\Rightarrow f$ is a *quotient map* ⟨exc. 22;2 p. 144 ∈ [1]⟩

8 Connectedness and Compactness

8.1 Fundamental Calculus Theorems depending on Compactness and Connectedness

- *Intermediate value theorem*: If $f : [a, b] \to \mathbb{R}$ *continuous* and $r \in \mathbb{R}$ and $f(a) < r < f(b)$, then $\exists c \in [a, b] : f(c) = r$.
  Depends on the *connectedness* of $[a, b]$. ⟨p. 147 ∈ [1]⟩

- *Maximum value theorem*: If $f : [a, b] \to \mathbb{R}$ *continuous*, then $\exists c \in [a, b] : \forall x \in [a, b] : f(x) \leq f(c)$.
  Depends on the *compactness* of $[a, b]$. ⟨p. 147 ∈ [1]⟩

- *Uniform continuity theorem*: If $f : [a, b] \to \mathbb{R}$ *continuous*, then $\forall \epsilon > 0 : \exists \delta > 0 : \forall x_1, x_2 \in [a, b] : |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$.
  Depends on the *compactness* of $[a, b]$. ⟨p. 147 ∈ [1]⟩

8.2 Connected Spaces

Premises: A, B, C ***topological spaces***

- **A *separation* of** $A \overset{def}{=} U, V \underset{open}{\subset} A$, where $U, V \neq \emptyset$ and $U \cap V = \emptyset$ and $U \cup V = A$. ⟨p. 148 ∈ [1]⟩

- A **is *connected*** $\overset{def}{=}$ there does *not exist* any *separation* of A. ⟨p. 148 ∈ [1]⟩

- A is *connected* $\Leftrightarrow$ the *only subsets* of A that are *both open and closed* are $\emptyset$ and A. ⟨p. 148 ∈ [1]⟩

- **For** $B \underset{subspace}{\subset} A$**, a *separation* of** $B \overset{def}{=} U, V \underset{subset}{\subset} B$, where U, V *nonempty* and *disjoint* and $U \cup V = B$ and *neither U nor V contains* a *limit point* of *the other* set. ⟨thm. 23.1 p. 148 ∈ [1]⟩

- **For** $B \underset{subspace}{\subset} A$**,** B **is *connected*** $\overset{def}{=}$ there does *not exist* any *separation* of B. ⟨thm 23.1 p. 148 ∈ [1]⟩

- Examples of *separations*:

  1. $]-1, 0[\cup]0, 1[$ as a *subspace* of $\mathbb{R}$. ⟨ex. 2 p. 149 ∈ [1]⟩
  2. Any *subspace* of $\mathbb{Q}$ *bigger* than a *one-element* set. ⟨ex. 4 p. 149 ∈ [1]⟩

- Examples of *connected spaces*:

  1. A *2-point set* with *indiscrete topology* ⟨ex. 1. p. 149 ∈ [1]⟩
  2. $[-1, 1]$ ⟨ex. 3. p. 149 ∈ [1]⟩

- If U, V form a *separation* of A and B is a *connected subspace* of A, then: B lies *entirely within either U or V* ⟨lemma 23.2 p. 149 ∈ [1]⟩

- A *collection* A_α of *connected subspaces* of A with *a point in common* $\Rightarrow$ $\cup A_\alpha$ is *connected* ⟨thm. 23.3 p. 159 ∈ [1]⟩

- Let B be a *connected subspace* of A. If $B \subset C \subset \overline{B}$, then C is *also connected.* ⟨thm. 23.4 p. 150 ∈ [1]⟩
  I.e. if C is formed by *adding some* or *all* of B's *limit points* to B, then C is *connected.*

- The *image* of a *connected space* under *continuous map* is *connected.* ⟨thm. 23.5 p. 150 ∈ [1]⟩

- *Finite cartesian products* of *connected spaces* are *connected.* ⟨thm. 23.6 p. 150 ∈ [1]⟩
  E.g.: 1) $\mathbb{R}^\omega$ is *not connected* in the *box topology.* ⟨ex. 6 p. 151 ∈ [1]⟩
  2) $\mathbb{R}^\omega$ *is connected* in the *product topology.* ⟨ex. 7 p. 151 ∈ [1]⟩

- For an *arbitrarily indexed family* $\{A_\alpha\}_{\alpha \in J}$ of *connected spaces*, the *product space* $\prod_{\alpha \in J} A_\alpha$ is *connected.* ⟨exc. 23;10 p. 152 ∈ [1]⟩

- **A space A is *totally disconnected*** $\overset{def}{=}$ the *only connected subspaces* are *one-point sets.* ⟨exc. 23;5 p. 152 ∈ [1]⟩

8.3 Connected Subspaces of the Real Line

- **A *linear continuum* L** $\stackrel{def}{=}$ ⟨p. 153 ∈ [1]⟩
  a *set* L with the *ordering properties* of $\mathbb{R}$. I.e.:
  1. L is *simply ordered*
  2. L has *more than one element*
  3. L has the *least upper bound* property
  4. $\forall x, y \in L : x < y \Rightarrow \exists z \in L : x < z < y$
- If L is a *linear continuum* in the *order topology*, then
  L *is connected* and so are *rays* and *intervals* in L. ⟨thm. 24.1 p. 153 ∈ [1]⟩
- For $B \underset{subspace}{\subset} A$, **$B$ is a *convex subspace* of A** $\stackrel{def}{=}$
  $\forall a, b \in B, a < b : [a, b] \subset B$. ⟨p. 153 ∈ [1]⟩
- R is *connected* and so are *intervals* and *rays* in $\mathbb{R}$. ⟨cor. 24.2 p. 154 ∈ [1]⟩
- *Intermediate value theorem*: Let $f : A \to B$ be a *continuous* map, where A is *connected* and B is an *ordered set* in the *order topology*.
  If $a, b \in A$, $r \in B$ and $f(a) < r < f(b)$, then $\exists c \in A : f(c) = r$.
  ⟨thm. 24.3 p. 154 ∈ [1]⟩
- Examples of *linear continua*: ⟨ex. 1, 2 p. 155 ∈ [1]⟩
  1. The *ordered square* is a *linear continuum*.
  2. If A is a *well-ordered set*, then: $A \times [0, 1[$ is a *linear continuum*.
- **A *path* in A from $x \in A$ to $y \in A$** $\stackrel{def}{=}$
  a *continuous map* $f : [a, b] \to A$, where
  $a, b \in \mathbb{R}, [a, b] \subset \mathbb{R}, f(a) = x, f(b) = y$. ⟨p. 155 ∈ [1]⟩
- **A space A is *path connected*** $\stackrel{def}{=}$
  $\forall a, b \in A$: there *exists* a *path in A* from a to b. ⟨p. 155 ∈ [1]⟩
- A space A is *path connected* $\Rightarrow$ A is *connected*
  (but the *converse* does *not* hold in general). ⟨p. 155 ∈ [1]⟩
- The *image* of a *path connected space* under a *continuous map* is *path connected*. ⟨ex. 5 p. 156 ∈ [1]⟩
- Examples of *path connected spaces*:
  1. *Open* and *closed balls* in $\mathbb{R}^n$ are *path connected*. ⟨ex. 3 p. 156 ∈ [1]⟩
  2. The *punctured euclidean plane* $\mathbb{R}^n \backslash \{\vec{0}\}$ is *path connected* for $n > 1$.
     ⟨ex. p. 156 ∈ [1]⟩
  3. The *unit sphere surface* S^{n-1} in $\mathbb{R}^n$ defined by $S^{n-1} = \{\vec{x} \mid ||\vec{x}|| = 1\}$ for $n > 1$ is *path connected*. ⟨ex. 5 p. 156 ∈ [1]⟩
- Examples of *non-path connected spaces*:
  1. The *ordered square* I_o^2 is *connected* but *not path connected*.
     ⟨ex. 6 p. 156 ∈ [1]⟩
  2. The *closure* $\bar{S}$ of the *Topologist's Sine Curve*
     $S = \{(x, \sin(1/x) \mid 0 < x \leq 1\}$ is *connected* but *not path connected*.
     ⟨ex. 7 p. 156-157 ∈ [1]⟩

8.4 Components and Local Connectedness

- **The *components* / *connected components*** of a *top. space* $A \overset{def}{=}$ the *equivalence classes* of the *equivalence relation* $\sim$ where $x \sim y$ means that there *exists* a *connected subspace* of A *containing* x and y. ⟨p. 159 ∈ [1]⟩

- The *components* of a *topological space* $A \equiv$ *connected disjoint subspaces* of A whose *union* is A, such that *each non-empty connected subspace* of A *intersects only one* of them. ⟨thm. 25.1 p. 159 ∈ [1]⟩

- **The *path components*** of a *topological space* $A \overset{def}{=}$ the *equivalence classes* of the *equivalence relation* $\sim$ where $x \sim y$ means that there *exists* a *path* in A from x to y. ⟨p. 160 ∈ [1]⟩

- The *path components* of a *topological space* $A \equiv$ *path connected disjoint subspaces* of A whose *union* is A, such that *each non-empty path connected subspace* of A *intersects only one* of them. ⟨thm. 25.2 p. 160 ∈ [1]⟩

- Each *component* is *closed* in a *topological space* A. ⟨p. 160 ∈ [1]⟩

- If a *topological space* A only has *finitely many components*, then *each component* is *open* in A. ⟨p. 160 ∈ [1]⟩

- Warning: *Path components* need *neither* be *open nor closed*. ⟨p. 160 ∈ [1]⟩

- A *topological space* A is ***locally connected* at** $x \in A \overset{def}{=}$ for *every neighborhood* U of x, there is a *connected neighborhood* V of x, where $V \subset U$. ⟨p. 161 ∈ [1]⟩

- A *topological space* is ***locally connected*** $\overset{def}{=}$ $\forall x \in A : A$ is *locally connected* at x. ⟨p. 161 ∈ [1]⟩

- A *topological space* A is ***locally path connected* at** $x \in A \overset{def}{=}$ for *every neighborhood* U of x, there is a *path connected neighborhood* V of x, where $V \subset U$. ⟨p. 161 ∈ [1]⟩

- A *topological space* is ***locally path connected*** $\overset{def}{=}$ $\forall x \in A : A$ is *locally path connected* at x. ⟨p. 161 ∈ [1]⟩

- A *topological space* A is *locally connected* $\Leftrightarrow$ ⟨thm. 25.3 p. 161 ∈ [1]⟩ for *every open set* U of A, *each component* of U is *open* in A.

- A *topological space* A is *locally path connected* $\Leftrightarrow$ ⟨thm. 25.4 p. 161 ∈ [1]⟩ for *every open set* U of A, *each path component* of U is *open* in A.

- For *topological space* A: *Each path component* of A *lies in* a *component* of A. ⟨thm. 25.5 p. 161 ∈ [1]⟩

- For *topological space* A: If A is *locally path connected*, then the *components* and the *path components* of A are the *same*. ⟨thm. 25.5 p. 161 ∈ [1]⟩

9 Compact Spaces

Premises: A, B ***topological spaces***

- **A *collection* $\mathcal{A}$ if *subsets* of A *covers* A / is a *covering* of A** $\stackrel{def}{=}$ the *union* of the *elements* of $\mathcal{A}$ is *equal* to A (i.e. $\cup\mathcal{A} = A$). ⟨p. 164 ∈ [1]⟩
- **A *collection* $\mathcal{A}$ if *subsets* of A is an *open covering* of A** $\stackrel{def}{=}$ $\mathcal{A}$ *covers* A and $\forall A_i \in \mathcal{A} : A_i \underset{open}{\subset} A$. ⟨p. 164 ∈ [1]⟩
- A **is *compact*** $\stackrel{def}{=}$ ⟨p. 164 ∈ [1]⟩ *every open covering* $\mathcal{A}$ of A has a *finite subcollection* that *also covers* A.
- For $B \underset{subspace}{\subset} A$: **A *collection* $\mathcal{A}$ of *subsets* of A *covers* B** $\stackrel{def}{=}$ $B \subset \cup\mathcal{A}$. ⟨p. 164 ∈ [1]⟩
- For $B \underset{subspace}{\subset} A$: B is *compact* $\Leftrightarrow$ *every covering* of B by *sets open* in A has a *finite subcollection covering* B. ⟨lemma 26.1 p. 164 ∈ [1]⟩
- For $B \underset{subspace}{\subset} A$: If A *compact* and $B \underset{closed\ subspace}{\subset} A$, then B *compact.* ⟨thm. 26.2 p. 165 ∈ [1]⟩
- For $B \underset{subspace}{\subset} A$: If A *Hausdorff* and B *compact*, then B *closed.* ⟨thm. 26.3 p. 165 ∈ [1]⟩
- For $B \underset{subspace}{\subset} A$: If A *Hausdorff* and B *compact* and $x_0 \notin B$, then there *exist disjoint open sets* U and V of A *containing* x_0 and B respectively. ⟨lemma 26.4 p. 166 ∈ [1]⟩

9.1 Compact Spaces as Tables

Premises: A, B ***topological spaces***, $\mathcal{A}$ **is a *collection* of *subsets* of** A.

A	B	property		equivalent
		$\mathcal{A}$ *covers* A a.k.a. $\mathcal{A}$ is a *covering* of A	$\overset{def}{=}$	$\cup\mathcal{A} = A$ ⟨p. 164 ∈ [1]⟩
		$\mathcal{A}$ is a *open covering* of A	$\overset{def}{=}$	$\mathcal{A}$ *covers* A and *all elements* in $\mathcal{A}$ are *open* ⟨p. 164 ∈ [1]⟩
	$B \underset{subspace}{\subset} A$	$\mathcal{A}$ *covers* B	$\overset{def}{=}$	$B \subset \cup\mathcal{A}$ ⟨p. 164 ∈ [1]⟩
		A *compact*	$\overset{def}{=}$	*every open covering* $\mathcal{A}$ of A has a *finite subcollection* that *also covers* A ⟨p. 164 ∈ [1]⟩
compact	$B \underset{subspace}{\subset} A$		$\Leftrightarrow$	*every covering* of B by *sets open* in A has a *finite subcollection covering* B ⟨lemma 26.1 p. 164 ∈ [1]⟩

Premises: A, B ***topological spaces***, $B \underset{subspace}{\subset} A$.

B	$\underset{subspace}{\subset}$	A	Misc.	$\Rightarrow$
	$\underset{closed\ subspace}{\subset}$	*compact*		B *compact* ⟨thm. 26.2 p. 165 ∈ [1]⟩
compact		*Hausdorff*		B *closed* ⟨thm. 26.3 p. 165 ∈ [1]⟩
compact		*Hausdorff*	$x_0 \notin B$	there *exist disjoint open sets* U and V of A *containing* x_0 and B respectively ⟨thm. 26.4 p. 165 ∈ [1]⟩

9.2 Compact Spaces Continued

Premises: A, B ***topological spaces***

- If A *compact* and f *continuous*, then $f(A)$ *compact*. ⟨thm. 26.5 p. 166 ∈ [1]⟩
- If A *compact*, B *Hausdorff* and $f : A \to B$ *bijective* and *continuous*, then f *homeomorphism*. ⟨thm. 26.6 p. 167 ∈ [1]⟩
- *Tychonoff theorem*:
  For $\{A_i\}_{i \in J}$ for *arbitrary* index set J, where $\forall i \in J : A_i$ *compact*:
  $\prod_{i \in J} A_i$ *compact*. ⟨thm. 26.7 p. 167, thm. 37.3 p. 224 ∈ [1]⟩
- The *tube lemma*: For *product space* $A \times B$, where B *compact*:
  If $N \underset{open}{\subset} A \times B$, where $x_0 \in A$ and the *slice* $x_0 \times B \subset N$, then
  N *contains some tube* $W \times B$ about $x_0 \times B$,
  where W is a *neighborhood* of x_0 in A. ⟨lemma 26.8 p. 168 ∈ [1]⟩
- Warning: The *tube lemma* does *not hold* if B is *not compact*!
  Example: B the y-axis of $\mathbb{R}^2$ and $N = \{(x, y) \mid |x| < \frac{1}{y^2+1}\}$ and $x_0 = 0$.
  Then N is *open*, $0 \times \mathbb{R} \subset N$, but N does *not contain any tube* about $0 \times \mathbb{R}$.
  ⟨ex. 7 p. 168 ∈ [1]⟩
- The *collection* $\mathcal{C}$ of *subsets* of the *topological space* A has
  the *finite intersection property* $\stackrel{def}{=}$ ⟨p. 169 ∈ [1]⟩
  for *every finite subcollection* $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$: $(C_1 \cap \cdots \cap C_n) \neq \emptyset$.
- For *topological space* A: A *compact* $\Leftrightarrow$
  for *every subcollection* $\mathcal{C}$ of *closed sets* in A having the *finite intersection property*: $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$ (i.e. the *intersection* of *all* $C \in \mathcal{C}$ is *non-empty*).
  ⟨thm. 26.9 p. 169 ∈ [1]⟩
- For *compact topological space* A:
  A *nested sequence* $C_1 \supset C_2 \supset \cdots C_n \supset C_{n+1} \supset \cdots$ of *closed sets* in A,
  where each $C_n \neq \emptyset$ automatically has the *finite intersection property*,
  so $\bigcap_{n \in \mathbb{Z}_+} C_n \neq \emptyset$. ⟨p. 170 ∈ [1]⟩

9.3 Basic Metric Stuff (Distance and Diameter)

- For (M, d) *metric space*, $C \subset M$, $C \neq \emptyset$:
  For each $x \in M$, **the *distance* from** x **to** A $\stackrel{def}{=}$
  $d(x, A) = \inf\{d(x, a) \mid a \in M\}$.
  ⟨p. 175 ∈ [1]⟩
- $d(x, A)$ is a *continuous function* of x. ⟨p. 175 ∈ [1]⟩
- For (M, d) *metric space*:
  The *diameter* $diam(C)$ **of a *bounded subset*** C **of** M $\stackrel{def}{=}$
  $diam(C) = \sup\{d(x_1, x_2) \mid x_1, x_2 \in C\}$.
  ⟨p. 175 ∈ [1]⟩

9.4 Compact Subspaces of the Real Line

- For *simply ordered set* A having the *least upper bound property*:
  In the *order topology*, each *closed interval* in A is *compact*.
  ⟨thm. 27.1 p. 127 ∈ [1]⟩

- *Every closed interval* in $\mathbb{R}$ is *compact*. ⟨cor. 27.2 p. 173 ∈ [1]⟩

- For $A \underset{subspace}{\subset} \mathbb{R}^2$: A *compact* $\Leftrightarrow$
  A *closed* and *bounded* in the *euclidean metric* d or the *square metric* ρ.
  ⟨thm. 27.3 p. 173 ∈ [1]⟩

- *Extreme Value Theorem*:
  For B *ordered set* in *order topology* and $f : A \to B$ *continuous*:
  If A *compact*, then there *exists points* $c, d \in A$ such that
  $\forall x \in B : f(c) \leq f(x) \leq f(d)$.
  I.e.: A *compact* $\Rightarrow \exists c, d \in A : \forall x \in B : f(c) \leq f(x) \leq f(d)$.
  ⟨thm. 27.4 p. 174 ∈ [1]⟩

- The *Lebesgue Number Lemma*:
  Let $\mathcal{A}$ be an *open covering* of the *metric space* (M, d).
  If M is *compact*, there is a $\delta > 0$ such that for *each subset* C of M having a *diameter less than* δ, there *exists* an *element* of $\mathcal{A}$ *containing* C.
  I.e.: M *compact* $\Rightarrow \exists \delta > 0 : \forall C \subset M, diam(C) < \delta : \exists A \in \mathcal{A} : C \subset A$.
  δ is a *Lebesgue number* for the *covering* $\mathcal{A}$. ⟨lemma 27.5 p. 175 ∈ [1]⟩

- For *metric spaces* $(A, d_A), (B, d_B)$:
  $f : A \to B$ **is *uniformly continuous*** $\stackrel{def}{=}$
  $\forall \epsilon > 0 : \exists \delta > 0 : \forall x_1, x_2 \in M : d_A(x_1, x_2) < \delta \Rightarrow d_B(f(x_1), f(x_2)) < \epsilon$.
  ⟨p. 176 ∈ [1]⟩

- *Uniform continuity theorem*: For $(A, d_A), (B, d_B)$ *metric spaces*:
  $f : A \to B$ *continuous* and A *compact* $\Rightarrow f$ *uniformly continuous*.
  ⟨thm. 27.6 p. 176 ∈ [1]⟩

- For *topological space* A: $x \in A$ **is an *isolated point* of** $A \stackrel{def}{=}$
  $\{x\}$ is *open* in A. ⟨p. 176 ∈ [1]⟩

- For *compact Hausdorff space* $A \neq \emptyset$:
  If A has *no isolated* points, then A is *uncountable*. ⟨thm. 27.7 p. 176 ∈ [1]⟩

- $\mathbb{R}$ is *uncountable*. ⟨p. 176-177 ∈ [1]⟩

- Every *closed interval* of $\mathbb{R}$ is *uncountable*. ⟨cor. 27.8 p. 177 ∈ [1]⟩

9.5 Limit Point Compactness

- For *topological space* A: A **is *limit point compact*** a.k.a. ***Fréchet compact*** a.k.a. **has the *Bolzano-Weierstrass* property** $\overset{def}{=}$ *every infinite subset* of A has a *limit point.* ⟨p. 178 ∈ [1]⟩

- For *topological space* A: A *compact* $\Rightarrow$ A *limit point compact* (but *not* conversely) ⟨thm. 28.1 p. 179 ∈ [1]⟩

- For *set* C, (x_n) a *sequence* of *points* in C: (x_{n_i}) **is a *subsequence* of** (x_n) $\overset{def}{=}$ $n_1 \leq n_2 \leq \cdots n_i \leq \cdots$ is an *increasing sequence* of *integers.* ⟨p. 179 ∈ [1]⟩

- For *topological space* A: A **is *sequentially compact*** $\overset{def}{=}$ *every sequence* of *points* of A has a *convergent subsequence.* ⟨p. 179 ∈ [1]⟩

- For *metrizable topological space* A, the *following are equivalent*:
  - A is *compact*
  - A is *limit point compact*
  - A is *sequentially compact*

  ⟨thm. 28.2 p. 179 ∈ [1]⟩

9.6 Local Compactness

- For *topological space* A: A **is *locally compact* at** $x \in A \overset{def}{=}$ there *exists* a $C \underset{compact\ subspace}{\subset} A$, where C *contains* a *neighborhood* of x. ⟨p. 182 ∈ [1]⟩
- For *topological space* A: A **is *locally compact*** $\overset{def}{=}$ $\forall x \in A : A$ is *locally compact* at x. ⟨p. 182 ∈ [1]⟩
- For *topological space* A: A *compact* $\Rightarrow$ A *locally compact.* ⟨p. 182 ∈ [1]⟩
- $\mathbb{R}^n$ is *locally compact*, but $\mathbb{R}^\omega$ is *not.* ⟨ex. 2 p. 182-183 ∈ [1]⟩
- Every *simply ordered set* A with the *least upper bound property* is *locally compact.* ⟨ex. 3 p. 183 ∈ [1]⟩
- For *topological space* A: A is *locally compact* and *Hausdorff* $\Leftrightarrow$ there *exists* a *topological space* B where:
  1. $A \underset{subspace}{\subset} B$.
  2. $B \backslash A$ consist of a *single point.*
  3. B is a *compact Hausdorff space.*

  If B and B' satisfies these properties, then there is a *homeomorphism* of B with B' that *equals the identity* map on A. ⟨thm. 29.1 p. 183 ∈ [1]⟩
  If A is *compact*: Then B is A with a *single isolated point added.*
  If A is *not compact*: Then $B \backslash A$ is a *limit point* of A so that $\bar{A} = B$. ⟨p. 184 ∈ [1]⟩
- For *topological spaces* A, B: B **is a *compactification* of** $A \overset{def}{=}$ B is a *compact Hausdorff space* and $A \underset{subspace}{\subsetneq} B$ and $\bar{A} = B$. ⟨p. 185 ∈ [1]⟩
- B **is the *one-point compactification* of** $A \overset{def}{=}$ B is a *compactification* of A and $B \backslash A$ is a *single point.* ⟨p. 185 ∈ [1]⟩
- The *one-point compactification* of $\mathbb{R}$ is *homeomorphic* to a *circle.* ⟨ex. 4 p. 185 ∈ [1]⟩
- The *one-point compactification* of $\mathbb{R}^2$ is *homeomorphic* to the *sphere* S^2. ⟨ex. 4 p. 185 ∈ [1]⟩
- **The *Riemann Sphere*** a.k.a. **the *extended plane*** $\overset{def}{=}$ $\mathbb{C} \cup \{\infty\}$, when $\mathbb{R}^2$ is seen as $\mathbb{C}$. ⟨ex. 4 p. 185 ∈ [1]⟩
- For *Hausdorff space* A: A is *locally compact* $\Leftrightarrow$ $\forall x \in A : \forall U$ *neighborhood* of $x : \exists V$ *neighborhood* of $x : \overline{V}$ is *compact* and $\overline{V} \subset U$. ⟨thm. 29.2 p. 185 ∈ [1]⟩
- For A *locally compact Hausdorff*, $B \underset{subspace}{\subset} A$: $B \underset{open}{\subset} A$ or $B \underset{closed}{\subset} A \Rightarrow B$ *locally compact.* ⟨cor. 29.3 p. 185 ∈ [1]⟩
- For *topological space* A: A is *homeomorphic* to an *open subspace* of a *compact Hausdorff space* $\Leftrightarrow$ A *locally compact Hausdorff.* ⟨cor. 29.4 p. 185 ∈ [1]⟩

10 Nets

- Note: Nets do for *general topological spaces* what *sequences* to for *metrizable spaces.* ⟨p. 187 ∈ [1]⟩

- **A *directed set* J** $\overset{def}{=}$ J is a set with partial order $\preccurlyeq$ such that $\forall \alpha, \beta \in J : \exists \gamma \in J : \alpha \preccurlyeq \gamma$ and $\beta \preccurlyeq \gamma$. ⟨p. 187 ∈ [1]⟩
  That $\preccurlyeq$ is a *partial order* means that the following hold:

  1. $\forall \alpha : \alpha \preccurlyeq \alpha$
  2. $\alpha \preccurlyeq \beta \wedge \beta \preccurlyeq \alpha \Rightarrow \alpha = \beta$
  3. $\alpha \preccurlyeq \beta \wedge \beta \preccurlyeq \gamma \Rightarrow \alpha \preccurlyeq \gamma$

  However all theorems here also hold if we only require the following of $\preccurlyeq$:

  1. $\forall \alpha : \alpha \preccurlyeq \alpha$
  2. $\alpha \preccurlyeq \beta \wedge \beta \preccurlyeq \gamma \Rightarrow \alpha \preccurlyeq \gamma$

  Many mathematitians use the term *directed set* in this more general case. ⟨exc. 29;12 p. 188 ∈ [1]⟩

- Some *directed sets*: ⟨exc. 29;1 p. 187 ∈ [1]⟩

  1. Any *simply ordered set* with *relation* $\leq$.
  2. The *collection* of *all subsets* of a *set* S, *partially ordered by inclusion* (i.e.: $A \preccurlyeq B$ when $A \subset B$).
  3. The *collection* $\mathcal{A}$ of *subsets* of a *set* S which is *closed* under *finite intersections*, *partially ordered by reverse inclusion* (i.e.: $A \preccurlyeq B$ when $A \supset B$).
  4. The *collection* of *all closed subsets* of a *topological space* A, *partially ordered by inclusion.*

- For *directed set* J and $K \subset J$: **K is *confinal* in J** $\overset{def}{=}$ for each $\alpha \in J$, there exists $\beta \in K$ such that $\alpha \preccurlyeq \beta$. ⟨exc. 29;2 p. 187 ∈ [1]⟩

- For *directed set* J and $K \subset J$: K is *confinal* in $J \Rightarrow$ K *directed set.* ⟨exc. 29;2 p. 187 ∈ [1]⟩

- For *topological space* A: **A *net* in A** $\overset{def}{=}$ a *function* $f : J \to A$ from a *directed set* J into A.
  If $\alpha \in J$, we usually *denote* $f(\alpha)$ by a_α.
  We *denote* the *net* f itself by $(a_\alpha)_{\alpha \in J}$ or merely (a_α) if the index is understood. ⟨exc. 29;3 p. 187 ∈ [1]⟩

- **The *net* (a_α) converges to $a \in A$** (written $a_\alpha \to a$) $\overset{def}{=}$ for *each neighborhood* U of a, there exists $\alpha \in J$ such that $\alpha \preccurlyeq \beta \Rightarrow a_\beta \in U$. ⟨exc. 29;3 p. 187 ∈ [1]⟩

- For *topological spaces* A, B: If $(x_\alpha)_{\alpha \in J} \to x$ in A and $(y_\alpha)_{\alpha \in J} \to y$ in B, then $(x_\alpha \times y_\alpha) \to x \times y$ in $A \times B$. ⟨exc. 29;4 p. 187 ∈ [1]⟩

- For *Hausdorff space* A: *A net* in A *converges* to *at most one point.* ⟨exc. 29;5 p. 187 ∈ [1]⟩

- For *topological space* A, (Error: $S \in A$?) $S \subset A$: $x \in \overline{S} \Leftrightarrow$ there *exists* a *net of points* of S *converging* to x. ⟨exc. 29;6 p. 187 ∈ [1]⟩

- For *topological spaces* A, B, $f : A \to B$: f *continuous* $\Leftrightarrow$ for *every convergent net* (x_α) in A converging to, say x, the net $(f(x_\alpha))$ converges to $f(x)$. ⟨exc. 29;7 p. 188 ∈ [1]⟩

- For *net* $f : J \to A$ in *topological space* A and $f(\alpha) = x_\alpha$:
  A *subnet* $f \circ g$ **of** $(x_\alpha) \overset{def}{=}$
  the *composite function* $f \circ g : K \to A$, where
  K is a *directed set* and $g : K \to J$ is a *function* such that:

  1. $i \preccurlyeq j \Rightarrow g(i) \preccurlyeq g(j)$
  2. $g(K)$ is *confinal* in J

  ⟨exc. 29;8 p. 188 ∈ [1]⟩

- If (x_α) *converges* to x, then so does *any subnet*. ⟨exc. 29;8 p. 188 ∈ [1]⟩

- For a *net* $(x_\alpha)_{\alpha \in J}$ in A:
  x **is an *accumulation point* of the net** $(x_\alpha) \overset{def}{=}$
  for *each neighborhood* U of x, the set of those α for which $x_\alpha \in U$ is *confinal* in J. ⟨exc. 29; 9 p. 188 ∈ [1]⟩

- The *net* (x_α) has x as an *accumulation point* $\Leftrightarrow$ *some subnet* of (x_α) *converges* to x. ⟨exc. 29;9 p. 188 ∈ [1]⟩

- A is *compact* $\Leftrightarrow$ *every net* in A has a *convergent subnet*. ⟨exc. 29;10 p. 188 ∈ [1]⟩

- For *topological group* G, $A, B \underset{subset}{\subset} G$:
  If $A \underset{closed}{\subset} G$ and B *compact*, then $A \cdot B$ is *closed* in G. ⟨exc. 29;11 p. 188 ∈ [1]⟩

11 Countability and Separation Axioms

11.1 Needed Definitions for Countability

Premises: A, B ***topological spaces***.

- A **has a *countable basis* at** $x \in A \overset{def}{=}$ there *exists* a *countable collection* $\mathcal{B}$ of *neighborhoods* of x such that *each neighborhood* of x *contains at least one* of the elements of $\mathcal{B}$. ⟨p. 190 ∈ [1]⟩
- $B \subset A$ **is *dense* in** $A \overset{def}{=} \overline{B} = A$. ⟨p. 191 ∈ [1]⟩

11.2 Countability Axioms

Premises: A, B ***topological spaces***.

- A ***Satisfies* the *first countability axiom***
  a.k.a. A ***is first countable*** $\overset{def}{=}$
  $\forall x \in A : A$ has a *countable basis* at x. ⟨p. 190 ∈ [1]⟩
- A ***Satisfies* the *second countability axiom***
  a.k.a. A ***is second countable*** $\overset{def}{=}$
  A has a *countable basis* for it's *topology*. ⟨p. 190 ∈ [1]⟩
- A **is a *Lindelöf space*** $\overset{def}{=}$
  Every open covering of A has a *countable subcollection covering* A.
  ⟨p. 192 ∈ [1]⟩
- A **is*separable*** (which has nothing to do with a *separation* of A) $\overset{def}{=}$
  There *exists* a *countable subset* of A which is *dense* in A.
  ⟨p. 192 ∈ [1]⟩

11.3 Theorems About Countability Axioms

Premises: A, B ***topological spaces***.

- Every *metrizable space* satisfies the *first countability axiom.* ⟨p. 190 ∈ [1]⟩
- For $B \subset A$:
  If there *exist* a *sequence of points* in B *converging* to x, then $x \in \overline{B}$.
  The *converse holds* if A is *first-countable.* ⟨thm. 30.1 a) p. 190 ∈ [1]⟩
- For $f : A \to B$: If f is *continuous*, then for *every convergent sequence* $x_n \to x$ in A, the sequence $f(x_n)$ converges to $f(x)$.
  The *converse holds* if A is *first-countable.* ⟨thm. 30.1 b) p. 190 ∈ [1]⟩
- *A first countable $\Rightarrow$ A second countable.* ⟨p. 190 ∈ [1]⟩
- A *metrizable space* is *not necessarily second countable.* ⟨p. 190 ∈ [1]⟩
- $\mathbb{R}^\omega$ and $\mathbb{R}^n$ are *second countable.* ⟨ex. 1 p. 190 ∈ [1]⟩
- $\mathbb{R}^\omega$ with the *uniform topology* is *first countable* but *not second countable.* ⟨p. 190 ∈ [1]⟩
- A *subspace* of a *first countable space* is *first countable.* ⟨thm. 30.2 p. 191 ∈ [1]⟩
- A *subspace* of a *second countable space* is *second countable.* ⟨thm. 30.2 p. 191 ∈ [1]⟩
- A *countable product* of *first countable spaces* is *first countable.* ⟨thm. 30.2 p. 191 ∈ [1]⟩
- A *countable product* of *second countable spaces* is *second countable.* ⟨thm. 30.2 p. 191 ∈ [1]⟩
- A is a *second countable topological space* $A \Rightarrow A$ is *Lindelöf.* ⟨thm. 30.3 a) p. 191 ∈ [1]⟩
- A is a *second countable topological space* $A \Rightarrow A$ is *separable.* ⟨thm. 30.3 b) p. 191 ∈ [1]⟩
- For *metrizable space* A: *A Lindelöf $\Leftrightarrow$ A second countable $\Leftrightarrow$ A separable.* ⟨p. 192 ∈ [1]⟩
- $\mathbb{R}_l$ is *first countable*, *Lindelöf* and *separable* but *not second countable.*
  I.e. $\mathbb{R}_l$ satisfies *all countability axioms except second countability.* ⟨ex. 3 p. 192 ∈ [1]⟩
- A *product space* of 2 *Lindelöf spaces* need *not* be *Lindelöf.* ⟨ex. 4 p. 193 ∈ [1]⟩
- A *subspace* of a *Lindelöf space* need *not* be *Lindelöf.* ⟨ex. 5 p. 193 ∈ [1]⟩
- **The *Sorgenfrey plane*** $\mathbb{R}_l^2 \overset{def}{=} \mathbb{R}_l \times \mathbb{R}_l$.
  $\mathbb{R}_l^2$ is *not Lindelöf.* ⟨ex. 5 p. 193 ∈ [1]⟩

11.4 Separation Axioms

Premises: A ***topological space***.

- (T_2) A **is *Hausdorff*** $\stackrel{def}{=}$
  $\forall x, y \in A, x \neq y$: there *exist disjoint open sets containing* x and y respectively. ⟨p. 195 ∈ [1]⟩

- For *topological space* A where *one-point sets* are *closed* in A:
  (T_3) A **is *regular*** $\stackrel{def}{=}$
  for *each pair* consisting of a *point* x and a *closed set* B *disjoint from* x,
  there *exist disjoint open sets containing* x and B, respectively. ⟨p. 195 ∈ [1]⟩

- For *topological space* A where *one-point sets* are *closed* in A:
  (T_4) A **is *normal*** $\stackrel{def}{=}$
  for *each pair* B, C of *disjoint closed sets* in A,
  there *exist disjoint open sets containing* B and C, respectively. ⟨p. 195 ∈ [1]⟩

- For *topological space* A where *one-point sets* are *closed* in A:
  A is *regular* $\Leftrightarrow$
  $\forall x \in A : \forall U$ *neighborhood* of $x : \exists V$ *neighborhood* of $x : \overline{V} \subset U$.
  ⟨lemma 31.1a p. 196 ∈ [1]⟩

- For *topological space* A where *one-point sets* are *closed* in A:
  A is *normal* $\Leftrightarrow$
  $\forall B \underset{closed}{\subset} A : \forall U \underset{open}{\subset} A$, where $B \subset U : \exists V \underset{open}{\subset} A$, where $B \subset V : \overline{V} \subset U$.
  ⟨lemma 31.1b p. 196 ∈ [1]⟩

- A *normal* $\Rightarrow$ A *regular*. ⟨p. 195 ∈ [1]⟩

- A *regular* $\Rightarrow$ A *Hausdorff*. ⟨p. 195 ∈ [1]⟩

- A *subspace* of a *Hausdorff space* is *Hausdorff*. ⟨thm. 31. 2a p. 196 ∈ [1]⟩

- A *product space* of *Hausdorff spaces* is *Hausdorff*. ⟨thm. 31. 2a p. 196 ∈ [1]⟩

- A *subspace* of a *regular space* is *regular*. ⟨thm. 31. 2a p. 196 ∈ [1]⟩

- A *product space* of *regular spaces* is *regular*. ⟨thm. 31. 2a p. 196 ∈ [1]⟩

- Warning: *Subspaces* or *products* of *normal spaces* need *not* be *normal*.
  ⟨p. 196 ∈ [1]⟩

- $\mathbb{R}_K$ is *Hausdorff* but *not regular*. ⟨ex. 1 p. 197 ∈ [1]⟩

- $\mathbb{R}_l$ is *normal*. ⟨ex. 2 p. 198 ∈ [1]⟩

- The *Sorgenfrey plane* $\mathbb{R}_l^2$ is *not normal*. ⟨ex. 3 p. 198 ∈ [1]⟩

12 Normal Spaces

Premises: A ***topological space.***

- Every *regular space* with a *countable basis* is *normal.* ⟨thm. 32.1 p. 200 ∈ [1]⟩
- Every *metrizable space* is *normal.* ⟨thm. 32.2 p. 202 ∈ [1]⟩
- Every *compact Hausdorff space* is *normal.* ⟨thm. 32.3 p. 202 ∈ [1]⟩
- Every *order topology* is *normal.* ⟨thm. 32.4, ex. 39 p. 202 ∈ [1]⟩
- If J is *uncountable*, the *product space* $\mathbb{R}^J$ is *not normal.* ⟨ex. 1 p. 203 ∈ [1]⟩
- $S_\Omega \times \overline{S_\Omega}$ is *not normal* ⟨ex. 2 p. 203 ∈ [1]⟩
  but is *completely regular.* ⟨ex. 1 p. 212 ∈ [1]⟩
- **(T_5) A is *completely normal*** $\overset{def}{=}$
  every subspace of A is *normal.* ⟨exc. 32;6 p. 205 ∈ [1]⟩
- *The Urysohn Lemma*:
  For *normal topological space* A and B, C *disjoint closed subsets* of A:
  If $[a, b]$ is a *closed interval* of the *real line*, then
  there exists a *continuous map* $f : A \to [a, b]$ such that
  $(\forall x \in B : f(x) = b)$ and $(\forall x \in C : f(x) = c)$. ⟨thm. 33.1 p. 207 ∈ [1]⟩
- The *converse* of the *Urysohn Lemma* is *trivial.* ⟨p. 211 ∈ [1]⟩
- For $B, C \subset A$:
  A and B can be *separated by a continuous function* $\overset{def}{=}$
  there *exist* a *continuous function* $f : A \to [0, 1]$ such that
  $f(A) = \{0\}$ and $f(B) = \{1\}$. ⟨p. 211 ∈ [1]⟩
- **($T_{3\frac{1}{2}}$) A is *completely regular*** $\overset{def}{=}$
  one-point sets are *closed* in A and
  for *each point* x_0 and *each closed set* C where $x_0 \notin C$, then
  there *exists* a *continuous function* $f : A \to [0, 1]$ such that
  $f(x_0) = 1$ and $f(C) = \{0\}$. ⟨p. 211 ∈ [1]⟩
- *A normal $\Rightarrow$ A completely regular.* ⟨p. 211 ∈ [1]⟩
- *A completely regular $\Rightarrow$ A regular.* ⟨p. 211 ∈ [1]⟩
- *Subspaces* of *completely regular spaces* are *completely regular.*
  ⟨thm. 33.2 p. 211 ∈ [1]⟩
- *Products* of *completely regular spaces* are *completely regular.*
  ⟨thm. 33.2 p. 211 ∈ [1]⟩
- **A is *perfectly normal*** $\overset{def}{=}$
  A is *normal* and *every closed set* in A is a G_δ-set in A. ⟨exc. 33;6 p. 213 ∈ [1]⟩
- *Urysohn metrization theorem*:
  Every *regular space* A with a *countable basis* is *metrizable.*
  ⟨thm. 34.1 p. 215 ∈ [1]⟩

- *Imbedding Theorem*:
  For *topological space* A where *one-point sets* are *closed*:
  If $\{f_\alpha\}_{\alpha \in J}$ is an *indexed family* of *continuous* functions $f_\alpha : A \to \mathbb{R}$ *satisfying* that for *each point* $x_0 \in A$ and *each neighborhood* U if x_0, there is an *index* α such that f_α is *positive* at x_0 and *vanishes outside* U. Then the *function* $F : A \to \mathbb{R}^J$ defined by $F(x) = (f_\alpha(x))_{\alpha \in J}$ is an *embedding* of A into $\mathbb{R}^J$.
  If f_α maps into $[0,1]$ for each α, then F *imbeds* A in $[0,1]^J$.
  ⟨thm. 34.2 p. 217 ∈ [1]⟩
  $\{f_\alpha\}_{\alpha \in J}$ is said to *separate points from closed sets* in A. ⟨p. 218 ∈ [1]⟩

- A is *completely regular* $\Leftrightarrow$
  A is *homeomorphic* to a *subspace* of $[0,1]^J$ for *some* J.
  ⟨thm. 34.3 p. 218 ∈ [1]⟩

12.1 Teitze Extension Theorem

Premises: A ***normal space***, $B \underset{closed\ subspace}{\subset} A$.

- Any *continuous map* $f : B \to [a,b]$ may be *extended* to a *continuous map* $f' : A \to [a,b]$, where $[a,b]$ is a *closed interval* of $\mathbb{R}$.
  ⟨thm. 35.1a p. 219 ∈ [1]⟩

- *Teitze extension theorem*: Any *continuous map* $f : B \to \mathbb{R}$ may be *extended* to a *continuous map* $f' : A \to \mathbb{R}$. ⟨thm. 35.1b p. 219 ∈ [1]⟩

13 Imbeddings of Manifolds

Premises: A ***topological space***.

- **An** m-***manifold*** $A \overset{def}{=}$
  a *Hausdorff space* A with a *countable basis* such that *each point* $x \in A$ has a *neighborhood* that is *homeomorphic* with an *open subset* of $\mathbb{R}^m$.
  ⟨p. 225 ∈ [1]⟩
  Note: A 1-*manifold* is often called a *curve* and a 2-*manifold* is often called a *surface.* ⟨p. 225 ∈ [1]⟩

- **The *support* of a *function*** $\phi : A \to \mathbb{R} \overset{def}{=}$
  $\overline{\phi^{-1}(\mathbb{R}\backslash\{0\})}$. Thus if $x \notin$ *support* of ϕ, there is *some neighborhood* of x where ϕ *vanishes.* ⟨p. 225 ∈ [1]⟩

- For *finite indexed open covering* $\{U_1, \dots, U_n\}$ of A:
  An *indexed family* of *continuous functions* $\phi_i : A \to [0,1]$ where $i \in \{1, \dots, n\}$ is **a *partition of unity dominated by*** $\{U_i\} \overset{def}{=}$

  1. $\forall i \in \{1, \dots, n\} :$ (support ϕ_i) $\subset U_i$
  2. $\forall x \in A : \sum_{i=1}^{n} \phi_i(x) = 1$

  ⟨p. 225 ∈ [1]⟩

- *Existence of finite partitions of unity* for *normal topological space* A:
  If $\{U_1, \dots, U_n\}$ is a *finite open covering* of A, then there *exists* a *partition of unity dominated by* $\{U_i\}$. ⟨thm. 36.1 p. 225 ∈ [1]⟩

- If A is a *compact* m*-manifold*, then A can be *imbedded* in $\mathbb{R}^N$ for *some positive integer* N. ⟨thm. 36.2 p. 226 ∈ [1]⟩

- For *set* S: If $\mathcal{A}$ is a *collection of subsets* of S with the *finite intersection property*, then there *exists* a collection $\mathcal{D}$ of *subsets* of S such that $\mathcal{D}$ *contains* $\mathcal{A}$, and $\mathcal{D}$ has the *finite intersection property* and *no other collection of subsets* of S that *properly contains* $\mathcal{D}$ has this property.
  I.e.: $\mathcal{D}$ is *maximal* w.r.t. the *finite intersection property.*
  ⟨lemma 37.1 p. 233 ∈ [1]⟩

- For a *set* S and a *collection* $\mathcal{D}$ of *subsets* of S that is *maximal* w.r.t. the *finite intersection property*:

  1. Any *finite intersection* of *elements* of $\mathcal{D}$ is an *element* of $\mathcal{D}$
  2. If C is a *subset* of S that *intersects every element* of $\mathcal{D}$, then C is an *element* of $\mathcal{D}$.

  ⟨lemma 37.2 p. 234 ∈ [1]⟩

14 Stone-$\check{C}$ech Compactification

Premises: A ***topological space***.

- A ***compactification*** B **of** $A \overset{def}{=}$
  a *compact Hausdorff space* B having A as a *subspace* such that $\overline{A} = B$.
  ⟨p. 237 ∈ [1]⟩
- **Two *compactifications*** B_1 **and** B_2 **of** A **are *equivalent*** $\overset{def}{=}$
  there *exists* a *homeomorphism* $h : B_1 \to B_2$ such that $\forall x \in A : h(x) = x$.
  ⟨p. 237 ∈ [1]⟩
- A *has* a *compactification* $B \Leftrightarrow A$ is *completely regular*. ⟨p. 237 ∈ [1]⟩
- For *Hausdorff space* C: If $h : A \to C$ is an *embedding* of A in C, then there *exists* a *corresponding compactification* B of A.
  B has the *property* that there *exists* an *embedding*
  $H : B \to C$ that *equals* h on A.
  The *compactification* B is *uniquely determined* up to *equivalence*.
  B is called the ***compactification induced* by the *imbedding*** h.
  ⟨lemma 38.1 p. 237 ∈ [1]⟩
- General note: There are *many compactifications.*
  Stone-Čech is in some sense the *maximal compactification* while the *one-point compactification* is in some sense the *minimal compactification.*
  ⟨p. 237-238 ∈ [1]⟩
- Compactification examples:
  - $[0, 1]$ is a *compactification* of $]0, 1[$,
    obtained by *adding one point* to *each end* of $]0, 1[$. ⟨ex. 2 p. 238 ∈ [1]⟩
  - The *unit circle* S^1 is *equivalent* to
    the *one-point compactification* of $]0, 1[$. ⟨ex. 1 p. 238 ∈ [1]⟩
  - Let $h :]0, 1[\to [-1, 1]^2$ be the map $h(x) = (x, \sin(1/x))$.
    The space $Y_0 = h(]0, 1[)$ is the *Topologist's Sine Curve.*
    (see ⟨ex. 7 paragraph 24 ∈ [1]⟩).
    The *imbedding* h gives a *compactification* of $]0, 1[$
    by *adding* a *point* and a *line segment* to $]0, 1[$. ⟨ex. 3 p. 238 ∈ [1]⟩
- *Compactification with extension condition*:
  For *completely regular space* A:
  There *exists* a *compactification* B of A having the property that *every bounded continuous map* $f : A \to \mathbb{R}$ *extends uniquely* to a *continuous map* of B into $\mathbb{R}$. ⟨thm. 38.2 p. 239 ∈ [1]⟩
- For *completely regular space* A, $S \subset A$,
  Hausdorff space C, $f : S \to C$ *continuous*:
  There is *at most one* extension of f to a *continuous* function $g : \overline{S} \to C$.
  ⟨lemma 38.3 p. 240 ∈ [1]⟩
- For *completely regular space* A, *compactification* B of A satisfying the *extension condition*, *compact Hausdorff space* C:
  Given *any continuous map* $f : A \to C$, the map f *extends uniquely* to a *continuous map* $g : B \to C$. ⟨thm. 38.4 p. 240 ∈ [1]⟩

- For *completely regular space* A:
  If B_1 and B_2 are *two compactifications* of A satisfying the *extension theorem*, then B_1 and B_2 are *equivalent.* ⟨thm. 38.5 p. 240 ∈ [1]⟩

- For *completely regular space* A:
  The *Stone-Čech compactification* $\beta(A)$ of $A \overset{def}{=}$ the *compactification* of A satisfying *the extension condition.*
  $\beta(A)$ is *characterized* by the fact that *any continuous map* $f : A \to C$ into a *compact Hausdorff space* C *extends uniquely* to a *continuous map* $g : \beta(A) \to C$. ⟨p. 241 ∈ [1]⟩

15 Questions

- Why does $]a, +\infty[\cup B$ and $]-\infty, a[\cup B$ form a *subbasis* for the *subspace topology* on B? (proof of thm. 16.4)

References

[1] James R. Munkres. *Topology, second edition*, Pearson Education (Prentice Hall?) 2000.