

# Topology Summary

Ánoq of the Sun, Hardcore Processing \*

July 1, 2004

## 1 Topology - The Creation

Premises:  $A, B$  sets

- **A topology on  $A$**   $\stackrel{def}{=}$  a collection  $\mathcal{T}$  of subsets of  $A$  where:
  - 1)  $\emptyset, A \in \mathcal{T}$
  - 2) For arbitrary index set  $I$ :  $(\forall i \in I : A_i \in \mathcal{T}) \Rightarrow \bigcup_{i \in I} A_i \in \mathcal{T}$
  - 3) For finite index set  $I$ :  $(\forall i \in I : A_i \in \mathcal{T}) \Rightarrow \bigcap_{i \in I} A_i \in \mathcal{T}$   
(p. 76 ∈ [1])
- **A topological space  $(A, \mathcal{T})$  with topology  $\mathcal{T}$**   $\stackrel{def}{=}$  a set  $A$  for which a topology  $\mathcal{T}$  has been specified. (p. 76 ∈ [1])
- For topological space  $A$  with topology  $\mathcal{T}$ :
  - $U$  is an **open set** of  $A$   $\stackrel{def}{=} U \subset A$  and  $U \in \mathcal{T}$  (p. 76 ∈ [1])
  - $U$  is a **neighborhood** of  $x$   $\stackrel{def}{=} U$  is an open set containing  $x$  (i.e.  $U$  open and  $x \in U$ ) (p. 96 ∈ [1])
  - Warning:** Some people say that  $U$  is a neighborhood of  $x$  if  $U$  contains an open set  $V$ , where  $x \in V$  (i.e.  $\exists V \subset U : x \in V$ ) (p. 97 ∈ [1])

### 1.1 Bases

- **A basis  $\mathcal{B}$  for a topology on  $A$**   $\stackrel{def}{=}$  a collection  $\mathcal{B}$  of subsets of  $A$  (called the *basis elements*) such that:
  - 1)  $\forall a \in A : \exists B \in \mathcal{B} : a \in B$
  - 2)  $\forall B_1, B_2 \in \mathcal{B} : a \in B_1 \cap B_2 \Rightarrow \exists B_3 \in \mathcal{B} : a \in B_3 \subset (B_1 \cap B_2)$   
(p. 78 ∈ [1])The *basis* of a given topology is not necessarily unique. (p. 80 ∈ [1])
- **The topology  $\mathcal{T}$  generated by the topology basis  $\mathcal{B}$ :**  
 $\mathcal{T} \stackrel{def}{=} \text{all } U \subset A \text{ such that: } \forall a \in U : \exists B \in \mathcal{B} : a \in B \subset U.$  (p. 78 ∈ [1])  
Prooving: Prove each of 1), 2), 3) in def. of topology separately like this: Use definition of generated topology and use the rules 1), 2) from topology basis as needed.
- $\mathcal{B}$  is a *basis* for the topology  $\mathcal{T} \Rightarrow$   
 $\mathcal{T}$  equals the collection of all unions of elements of  $\mathcal{B}$ . (lemma 13.1 p. 80 ∈ [1])
- Let  $(A, \mathcal{T})$  be a given topological space.  
If  $\mathcal{C} \subset \mathcal{T}$  such that:  $\forall U \in \mathcal{T} : \forall x \in U : \exists C \in \mathcal{C} : x \in C \subset U,$   
then  $\mathcal{C}$  is a *basis* for  $\mathcal{T}$ . (lemma 13.2 p. 80 ∈ [1])

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## 1.2 Subbases

- **A subbasis  $\mathcal{S}$  for a topology on  $A$**   $\stackrel{def}{=} \equiv$  a collection  $\mathcal{S}$  of subsets of  $A$ , where  $A = \bigcup_{S \in \mathcal{S}} S$  (p. 82 ∈ [1])  
(The *subbasis* of a *given topology* is *not necessarily unique*?)
- **The topology  $\mathcal{T}$  generated by the topology subbasis  $\mathcal{S}$ :**  
 $\mathcal{T} \stackrel{def}{=} \equiv$  all unions of finite intersections of elements of  $\mathcal{S}$ . (p. 82 ∈ [1])  
I.e. all  $U \subset A$  such that:  $\exists S_{ij} \in \mathcal{S} : U = \bigcup_{i \in I} \bigcap_{j \in J} S_{ij}$  for finite  $J$ .

## 2 Common and Well-Known

### 2.1 Common Well-Known Topologies

Premises:  $A, B$  sets

- **Discrete topology** on  $A \stackrel{def}{=} \mathcal{T}$  is all subsets of  $A$  (ex. 2 p. 77 ∈ [1])
- **Indiscrete / trivial topology** on  $A \stackrel{def}{=} \mathcal{T} = \{\emptyset, A\}$  (ex. 2 p. 77 ∈ [1])
- **Finite complement topology**  $\mathcal{T}_f$  on  $A \stackrel{def}{=} \mathcal{T}_f$  is a collection of subsets  $U \subset A$  such that  $A \setminus U$  is either finite or all of  $A$  (ex. 3 p. 77 ∈ [1])
- $\mathcal{T}_c$  on  $A \stackrel{def}{=} \mathcal{T}_c$  all subsets  $U \subset A$  such that  $A \setminus U$  is either countable or all of  $A$  (ex. 4 p. 77 ∈ [1])
- **Standard topology** on  $\mathbb{R} \stackrel{def}{=} \mathcal{T}$   
 $\mathcal{T}$  = generated by basis consisting of all open intervals on  $\mathbb{R}$ ,  
 i.e. : all  $]a, b[ = \{x \mid a < x < b\}$ . (p. 81 ∈ [1])  
 This is just the *order topology* on  $\mathbb{R}$ , with order relation  $<$ . (ex. 1 p. 85 ∈ [1])
- **Lower limit topology** on  $\mathbb{R} \stackrel{def}{=} \mathcal{T}$   
 $\mathcal{T}$  = generated by basis consisting of all intervals of the form  $[a, b[$ .  
 We call  $\mathbb{R}$  with this topology  $\mathbb{R}_l$ . (p. 82 ∈ [1])
- **K-topology** on  $\mathbb{R} \stackrel{def}{=} \mathcal{T}$  Let  $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ , then  
 $\mathcal{T}$  = generated by basis consisting of all sets of the forms  $]a, b[$  or  $]a, b[ \setminus K$ .  
 We call  $\mathbb{R}$  with this topology  $\mathbb{R}_K$ . (p. 82 ∈ [1])
- **Order topology**  $\stackrel{def}{=} \mathcal{T}$  generated by the *order topology basis* (p. 84 ∈ [1])
- **Product topology** on  $A \times B \stackrel{def}{=} \mathcal{T}$   
 the topology generated by the basis  $\mathcal{B}$  where  
 $\mathcal{B}$  is all sets  $U \times V$  such that  $U \stackrel{C}{open} A, V \stackrel{C}{open} B$ .  
 (the basis  $\mathcal{B}$  is not a topology though) (p. 86 ∈ [1])
- **Product topology** on  $\prod_{\alpha \in J} A_\alpha$ , where  $A_\alpha$  are *topological spaces*  $\stackrel{def}{=} \mathcal{T}_\alpha$   
 the topology generated by the *product topology subbasis*. (p. 114 ∈ [1])  
 We call  $\prod_{\alpha \in J} A_\alpha$  with this topology a **product space**.  
 Can be described as: all sets  $\prod_{\alpha \in J} U_\alpha$ , where  $(\forall \alpha \in J : U_\alpha \stackrel{C}{open} A_\alpha)$  and  
 $U_\alpha = A_\alpha$  except for finitely many  $\alpha$  (thm. 19.1 p. 115 ∈ [1])
- **Box topology** on  $\prod_{\alpha \in J} A_\alpha \stackrel{def}{=} \mathcal{T}$   
 The topology generated by the *box topology basis* (p. 113, 114 ∈ [1])  
 Can be described as: all sets  $\prod_{\alpha \in J} U_\alpha$ , where  $\forall \alpha \in J : U_\alpha \stackrel{C}{open} A_\alpha$  (thm. 19.1 p. 115 ∈ [1])
- **Subspace topology**  $\mathcal{T}_B$  on  $(A, \mathcal{T}) \stackrel{def}{=} \mathcal{T}$   
 For  $B \subset A : \mathcal{T}_B = \{B \cap U \mid U \in \mathcal{T}\}$ .  
 We have:  $(B, \mathcal{T}_B) \stackrel{C}{subspace} (A, \mathcal{T})$  (p. 88 ∈ [1])  
 Properties for  $(B, \mathcal{T}_B) \stackrel{C}{subspace} (A, \mathcal{T})$  :  
 1)  $U$  open in  $B$  a.k.a.  $U$  open relative to  $B \stackrel{def}{=} U \in \mathcal{T}_B$  (p. 89 ∈ [1])  
 2)  $U$  open in  $A$  a.k.a.  $U$  open relative to  $A \stackrel{def}{=} U \in \mathcal{T}$  (p. 89 ∈ [1])  
 3)  $U$  open in  $B$  and  $B$  open in  $A \Rightarrow U$  open in  $A$  (lemma 16.2 p. 89 ∈ [1])

- **Ordered square  $I_0^2$ , where  $I \stackrel{\text{def}}{=} \underset{\text{interval}}{\subset} \mathbb{R}$**  with *dictionary order* and *order topology* (ex. 3 p. 90 ∈ [1])

## 2.2 Very Concrete Well-Known Topologies

- $\mathbb{R} \times \mathbb{R}$  with the *dictionary order*. Possible bases: (ex. 2 p. 85 ∈ [1])
  - 1) The collection of intervals  $]a, b), (c, d[$  where  $a < c$  or  $(a = c \wedge b < d)$
  - 2) The collection of intervals  $]a, b), (c, d[$  where  $(a = c \wedge b < d)$
- $\mathbb{Z}_+$  with the *order topology* is the *discrete topology*. (ex. 3 p. 85 ∈ [1])
- The *order topology* for  $\{1, 2\} \times \mathbb{Z}_+$  with the *dictionary order* is *not* the *discrete topology* (ex. 4 p. 85 ∈ [1])

## 2.3 Common Well-Known Topology Bases

Premises:  $A, B$  sets

- The collection of *all circular regions* (*interiors* of circles) in  $\mathbb{R}^2$  (ex. 1 p. 78 ∈ [1])
- The collection of *all rectangular regions* (*interiors* of rectangles) in  $\mathbb{R}^2$  (ex. 2 p. 79 ∈ [1])  
This is a basis for the *standard topology* on  $\mathbb{R}^2$   
(the *product* of 2 *order topologies*:  $\mathbb{R} \times \mathbb{R}$ ) (ex. 1 p. 87 ∈ [1])
- The collection of *all one-point subsets* of *any set*  $A$ .  
This is the basis for the *discrete topology* (ex. 3 p. 79 ∈ [1])
- *Topologies* which also qualify as bases:  
*standard topology, K-topology, lower limit topology.* (p. 82 ∈ [1])
- **Basis for the *order topology* on  $A$ , where  $A$  is an *ordered set* with *simple order relation*  $< \stackrel{\text{def}}{=} \mathcal{B}$  is given by:**
  - 1) *All open intervals*  $]a, b[$  in  $A$
  - 2) If  $A$  has a *smallest element*  $a_0$ : Then also *all intervals*  $[a_0, b[$
  - 3) If  $A$  has a *largest element*  $b_0$ : Then also *all intervals*  $]a, b_0]$

(p. 84 ∈ [1])
- **Alternative basis  $\mathcal{B}$  for the *product topology* on  $A \times B$ :**  
If  $\mathcal{B}$  is *basis* for the *topology* on  $A$  and  $\mathcal{C}$  *basis* for the *topology* on  $B$ , then  $\mathcal{D} = \{(B, C) \mid B \in \mathcal{B} \wedge C \in \mathcal{C}\}$  is *basis* for the *topology* on  $A \times B$  (thm. 15.1 p. 86 ∈ [1])
- **Alternative basis  $\mathcal{B}$  for the *product topology* on  $\prod_{\alpha \in J} A_\alpha$ ,**  
where each  $A_\alpha$  is given by the basis  $\mathcal{B}_\alpha \stackrel{\text{def}}{=}$   
 $\mathcal{B} =$  *all sets* of the form  $\prod_{\alpha \in J} B_\alpha$ , where for  $\forall I \stackrel{\text{def}}{=} \underset{\text{finite}}{\subset} J$  we have that:  
 $\forall \alpha \in I : B_\alpha \in \mathcal{B}_\alpha$  and  $\forall \alpha \in J \setminus I : B_\alpha = A_\alpha$  (thm. 19.2 p. 116 ∈ [1])  
I.e.  $B_\alpha \in \mathcal{B}_\alpha$  for *finitely many*  $\alpha$  and  $B_\alpha = A_\alpha$  for the *other*  $\alpha$ .
- **Basis  $\mathcal{B}$  for the *box topology* on  $\prod_{\alpha \in J} A_\alpha \stackrel{\text{def}}{=}$**   
Let  $\{A_\alpha\}_{\alpha \in J}$  be an *indexed family* of *topological spaces*, the basis  $\mathcal{B}$  on the *product space*  $\prod_{\alpha \in J} A_\alpha$  is given by:  
 $\mathcal{B} =$  *all sets* of the form  $\prod_{\alpha \in J} U_\alpha$ , where  $\forall \alpha \in J : U_\alpha \stackrel{\text{def}}{=} \underset{\text{open}}{\subset} A_\alpha$  (p. 114 ∈ [1])

- **Alternative basis  $\mathcal{B}$  for the box topology on  $\prod_{\alpha \in J} A_\alpha$ ,**  
 where each  $A_\alpha$  is given by the basis  $\mathcal{B}_\alpha \stackrel{def}{=} \mathcal{B}$   
 $\mathcal{B} =$  all sets of the form  $\prod_{\alpha \in J} B_\alpha$ , where  $\forall \alpha \in J : B_\alpha \in \mathcal{B}_\alpha$  (thm. 19.2 p. 116  
 ∈ [1])
- **A basis  $\mathcal{B}_B$  for the subspace topology on  $(A, \mathcal{T})$ :**  
 For  $B \subset A$ , and  $\mathcal{B}$  a basis for  $(A, \mathcal{T})$ , then  
 $\mathcal{B}_B = \{B \cap Y \mid Y \in \mathcal{B}\}$ . (lemma 16.1 p. 89 ∈ [1])

## 2.4 Common Well-Known Topology Subbases

Premises:  $A, B$  sets

- **Subbasis for the order topology on  $A \stackrel{def}{=} \mathbb{R}$**   
 The open rays on  $A$ :  $]a, +\infty[ = \{x \mid x > a\}$  and  $] - \infty, a[ = \{x \mid x < a\}$  (p. 86 ∈ [1])
- **Subbasis for the product topology on  $A \times B \stackrel{def}{=} A \times B$**   
 $\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } A\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } B\}$ , where  
 $\pi_1 : A \times B \rightarrow A, \pi_2 : A \times B \rightarrow B$  are the projections (thm. 15.2 p. 88 ∈ [1])
- **Subbasis  $\mathcal{S}$  for the product topology on  $\prod_{\alpha \in J} A_\alpha \stackrel{def}{=} \prod_{\alpha \in J} A_\alpha$**   
 Let  $\mathcal{S}_\alpha = \{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \stackrel{C}{\subset} A_\alpha\}$ , then  
 $\mathcal{S} = \cup_{\alpha \in J} \mathcal{S}_\alpha$ , where  $\pi_\alpha : (\prod_{\alpha \in J} A_\alpha) \rightarrow A_\alpha$  are the projections (p. 114 ∈ [1])

## 2.5 Theorems About Well-Known Topologies

- Let  $B_\alpha \stackrel{C}{\subset} A_\alpha$  for all  $\alpha \in J$ . Then  $(\prod B_\alpha) \stackrel{C}{\subset} (\prod A_\alpha)$  if:  
 either both have the product topology or both have the box topology. (thm.  
 19.3 p. 116 ∈ [1])

### 3 Comparison of Topologies

$A, B$  sets,  $\mathcal{T}, \mathcal{T}'$  topologies on  $A$

- For  $\mathcal{T} \subset \mathcal{T}'$  we define:  
 $\mathcal{T}$  *coarser* / *smaller than*  $\mathcal{T}'$   
 $\mathcal{T}'$  *finer* / *larger than*  $\mathcal{T}$   
 Beware of the terms *weaker* / *stronger*! (some people have incompatible meanings for these terms)  
(p. 77 ∈ [1])
- For  $\mathcal{T} \subsetneq \mathcal{T}'$  we define:  
 $\mathcal{T}$  *strictly coarser* / *strictly smaller than*  $\mathcal{T}'$   
 $\mathcal{T}'$  *strictly finer* / *strictly larger than*  $\mathcal{T}$   
(p. 77 ∈ [1])
- $\mathcal{T}$  *comparable with*  $\mathcal{T}' \stackrel{def}{=} \text{either } \mathcal{T} \subset \mathcal{T}' \text{ or } \mathcal{T}' \subset \mathcal{T}$  (p. 77 ∈ [1])
- Let  $\mathcal{B}$  and  $\mathcal{B}'$  be *bases* for the *topologies*  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively, on  $A$ .  
 Then the following are *equivalent*:  
 1)  $\mathcal{T}'$  is *finer* than  $\mathcal{T}$   
 2)  $\forall a \in A : \forall B \in \mathcal{B}, a \in B : \exists B' \in \mathcal{B}' : a \in B' \subset B$   
(lemma 13.3 p. 81 ∈ [1])

#### 3.1 Comparison of Well-Known Topologies

Premises:  $A, B$  sets

- If  $A$  *subspace* of  $X$  and  $B$  *subspace* of  $Y$ , then  
 the *product topology* on  $A \times B$  is the *same as*  
 the topology  $A \times B$  *inherits* as a *subspace* of  $X \times Y$ . (lemma 16.3 p. 89 ∈ [1])
- **Warning:** Let  $A$  be an *ordered set* with *order topology*, and let  $B \subset A$ .  
 The *order relation* on  $A$  *restricted to*  $B$  makes  $B$  an *ordered set*. However:  
 The *resulting order topology* on  $B$  *in general*  $\neq$  the *topology that*  $B$  *inherits*  
 as a *subspace* of  $A$ . (p. 90 ∈ [1])  
 Examples:  
 1) The *topology of*  $[0, 1]$  as a *subspace* of  $\mathbb{R} = \text{order topology on } [0, 1]$  (ex. 1 p. 90 ∈ [1])  
 2) The *topology of*  $B = [0, 1] \cup \{2\}$  as a *subspace* of  $\mathbb{R} \neq \text{order topology on } B$  (ex. 2 p. 90 ∈ [1])  
 3) The *topology of*  $B = [0, 1] \times [0, 1]$  with *dictionary order* as a *subspace* of  $\mathbb{R} \times \mathbb{R} \neq$   
*order topology on*  $[0, 1] \times [0, 1]$  (ex. 3 p. 90 ∈ [1])
- **To avoid ambiguity:** When  $A$  is an *ordered set* in *order topology* and  
 $B \subset A$ , we *assume* that  $B$  is *given* the *subspace topology* (unless otherwise  
 stated). (p. 91 ∈ [1])
- If  $A$  is an *ordered set* with *order topology* and  $B \stackrel{C}{\text{convex subset}} B$ , then  
*order topology on*  $B = \text{topology } B \text{ inherited as a subspace of } A$ . (thm. 16.4  
 p. 91 ∈ [1])
- *standard topology*  $\subset$  *lower limit topology* (lemma 13.4 p. 82 ∈ [1])
- *standard topology*  $\subset$  *K-topology* (lemma 13.4 p. 82 ∈ [1])
- *lower limit topology* is *not comparable* to *K-topology* (lemma 13.4 p. 82 ∈ [1])
- *Product topology on*  $\mathbb{R}^J \subset$  *uniform topology on*  $\mathbb{R}^J \subset$  *box topology on*  $\mathbb{R}^J$ .  
 If  $J$  is *infinite*, they are *all different*. (thm. 20.4 p. 124 ∈ [1])

## 4 Topological Properties

- **Topological property**  $\stackrel{def}{=}$  property expressed *only via open sets* (p. 105 ∈ [1])

### 4.1 Closed Sets

Premises:  $(A, \mathcal{T})$  *topological space*

- **A subset  $U$  of  $A$  is closed**  $\stackrel{def}{=} U \setminus A$  *open*. (p. 93 ∈ [1])
- The following hold:
  - 1)  $\emptyset$  and  $A$  are *closed*
  - 2) *Arbitrary intersections of closed sets are closed*
  - 3) *Finite unions of closed sets are closed*
- For  $(B, \mathcal{T}_B) \stackrel{\subset}{\text{subspace}} (A, \mathcal{T})$  then the following hold:
  - 1)  $U$  *closed* in  $B \Leftrightarrow \exists V \stackrel{\subset}{\text{closed}} A : U = B \cap V$  (thm. 17.2 p. 94 ∈ [1])
  - 2)  $U$  *closed* in  $B$  and  $B$  *closed* in  $A \Rightarrow U$  *closed* in  $A$  (thm. 17.3 p. 95 ∈ [1])

### 4.2 Closure, Interior

Premises:  $(A, \mathcal{T})$  *topological space*

- **Interior of  $B \subset A$**   $\equiv \text{Int } B \stackrel{def}{=} \bigcup_{B_i \stackrel{\subset}{\text{open}} A, B_i \subset B} B_i$  (p. 95 ∈ [1])
- **Closure of  $B \subset A$**   $\equiv \text{Cl } B$  w.r.t.  $A \equiv \text{Cl}_A B \equiv \overline{B} \stackrel{def}{=} \bigcap_{B \subset B_i \stackrel{\subset}{\text{closed}} A} B_i$  (p. 95 ∈ [1])
- $\text{Int } A \subset A \subset \overline{A}$  (p. 95 ∈ [1])
- $A$  *open*  $\Rightarrow A = \text{Int } A$  (p. 95 ∈ [1])
- $A$  *closed*  $\Rightarrow A = \overline{A}$  (p. 95 ∈ [1])
- For  $(B, \mathcal{T}_B) \stackrel{\subset}{\text{subspace}} (A, \mathcal{T})$  and  $U \subset B$ , we have:
  - 1) **Warning:**  $\text{Cl}_B U \stackrel{\neq}{\text{in general}} \text{Cl}_A U$ . In such cases *we assume*  $\text{Cl } U = \text{Cl}_A U$  (p. 95 ∈ [1])
  - 2)  $\text{Cl}_B U = (\text{Cl}_A U) \cap B$  (thm. 17.4 p. 95 ∈ [1])
- For  $U \subset A$ , we have:
 
$$x \in \text{Cl}_A U \Leftrightarrow \forall V \in \mathcal{T}, x \in V : U \cap V \neq \emptyset$$
 (i.e. *every neighborhood of  $x$  intersects  $A$* ) (thm. 17.5 p. 96 ∈ [1])
- If  $\mathcal{T}$  is given by a basis  $\mathcal{B}$ , then:
 
$$x \in \text{Cl}_A U \Leftrightarrow \forall B \in \mathcal{B}, x \in B : B \cap U \neq \emptyset$$
 (thm. 17.5 p. 96 ∈ [1])

#### 4.2.1 Examples

- In the *finite complement topology* on  $A$ , the *closed sets* are:  
The set  $A$  itself and *all finite subsets* of  $A$ . (ex. 3 p. 93 ∈ [1])
- In the *discrete topology* on  $A$ :  
*All sets are open, so all sets are closed as well.* (ex. 4 p. 93 ∈ [1])
- Consider the *subspace*  $A = ]0, 1]$  of  $\mathbb{R}$ . The set  $U = ]0, \frac{1}{2}[$  is a *subset* of  $A$ .  
 $\text{Cl}_{\mathbb{R}} U = [0, \frac{1}{2}] \neq \text{Cl}_A U = [0, \frac{1}{2}] \cap A = ]0, \frac{1}{2}]$  (ex. 7 p. 97 ∈ [1])

### 4.3 Limit Points

Premises:  $(A, \mathcal{T})$  *topological space*

- **Limit point / cluster point / point of accumulation  $x$  of  $A$**   $\stackrel{def}{=}$   
Every neighborhood of  $x$  intersects  $A$  in some other point  $y \neq x \equiv$   
 $x \in (\text{Cl}_A(A \setminus \{x\}))$  (p. 97 ∈ [1])
- **Notation:** When  $x_n$  is sequence of points in  $A$  converging to  $x \in A$ ,  
we write  $x_n \rightarrow x$ ,  $x$  is the *limit* of  $x_n$ . (p. 100 ∈ [1])
- For a subset  $U \subset A$ :
  1. Let  $B'$  be the set of all limit points of  $B$ , then  $\text{Cl}_A B = B \cup B'$  (thm. 17.6 p. 97 ∈ [1])
  2.  $B$  closed  $\Leftrightarrow$  (all limit points of  $B$ )  $\stackrel{\subset}{\text{subset}} B$  (cor. 17.7 p. 98 ∈ [1])
- For  $\{A_\alpha\}$  an indexed family of spaces, let  $B_\alpha \subset A_\alpha$  for each  $\alpha$ .  
If  $\prod A_\alpha$  has either product topology or box topology, then  
 $\overline{\prod B_\alpha} = \prod \overline{B_\alpha}$ . (thm. 19.5 p. 116 ∈ [1])

#### 4.3.1 Hausdorff Spaces

Premises:  $(A, \mathcal{T})$  *Hausdorff space*

- $(A, \mathcal{T})$  **Hausdorff space**  $\stackrel{def}{=}$   
 $\forall x, y \in A, x \neq y$  : there exists disjoint neighborhoods  $U_1$  and  $U_2$  of  $x$  and  $y$  respectively. (p. 98 ∈ [1])
- Every finite point set in any Hausdorff space is closed. (thm. 17.8 p. 99 ∈ [1])
- A sequence of points of  $A$  converges to at most one point of  $A$ . (thm. 17.10 p. 99 ∈ [1])
- $(A, \mathcal{T})$  Hausdorff  $\Leftrightarrow$  the diagonal  $\Delta = \{(a, a) \mid a \in A\}$  is closed in  $A \times A$ .  
(exc. 17; 13 p. 101 ∈ [1])
- Well-known Hausdorff spaces:
  - The order topology of any simply ordered set (exc. 17;10 p. 101 ∈ [1])
  - A subspace of a Hausdorff space (exc. 17;12 p. 101 ∈ [1])
- If  $\forall \alpha \in J : A_\alpha$  Hausdorff, then  $\prod A_\alpha$  is Hausdorff  
in both the product topology and the box topology. (thm. 19.4 p. 116 ∈ [1])

#### 4.3.2 The $T_1$ axiom

Premises:  $(A, \mathcal{T})$  *topological space satisfying the  $T_1$  axiom*

- **The  $T_1$  axiom**  $\stackrel{def}{=}$  finite point sets are closed (weaker than Hausdorff property) (p. 99 ∈ [1])
- Let  $B \stackrel{\subset}{\text{subset}} A$ . Then the point  $x$  is a limit point of  $B \Leftrightarrow$   
every neighborhood of  $x$  contains infinitely many points of  $B$  (thm. 17.9 p. 99 ∈ [1])

## 5 Continuous Functions

### 5.1 Well-Known Functions

Premises:  $A, B$  *topological spaces*

- $f : A \rightarrow B$  **open map**  $\stackrel{def}{=} U \underset{open}{\subset} A \Rightarrow f(U) \underset{open}{\subset} B$  (exc. 4 p.92 ∈ [1])
- If  $U \underset{open}{\subset} X_i$  then,  
 $\pi_i^{-1}(U) = (X_1 \times X_2 \times \cdots \times U \times \cdots \times X_n) \underset{open}{\subset} (X_1 \times X_2 \times \cdots \times X_i \times \cdots \times X_n)$   
 (p. 87 ∈ [1])

### 5.2 Continuous Functions

Premises:  $A, B$  *topological spaces*

- $f : A \rightarrow B$  **continuous** (relative to the topologies on  $A$  and  $B$ )  $\stackrel{def}{=} \forall V \underset{open}{\subset} B : f^{-1}(V) \underset{open}{\subset} A$ . (p. 102 ∈ [1])  
 So *continuity depends both on function and topologies for  $A$  and  $B$ .*  
 Example (lower limit topology and identity function):  
 $id : \mathbb{R} \rightarrow \mathbb{R}_l$  is *not continuous*, but  $id : \mathbb{R}_l \rightarrow \mathbb{R}$  is *continuous*.
- $f : A \rightarrow B$  **continuous at  $a \in A$**   $\stackrel{def}{=} \forall V$  neighborhood of  $f(a) : \exists U$  neighborhood of  $a : f(U) \subset V$  (p. 104 ∈ [1])
- If *topology of  $B$  is given by the topology BASIS  $\mathcal{B}$* , then:  
 $(\forall B_i \in \mathcal{B} : f^{-1}(B_i) \underset{open}{\subset} A) \Rightarrow f : A \rightarrow B$  *continuous* (p. 103 ∈ [1])  
 Proving: Any open set  $V$  (a union of basis elements):  $V = \bigcup_{\alpha \in J} B_\alpha$ . So  $f^{-1}(V) = \bigcup_{\alpha \in J} (f^{-1}(B_\alpha))$
- If *topology of  $B$  is given by the topology SUBBASIS  $\mathcal{S}$* , then:  
 $(\forall S_i \in \mathcal{S} : f^{-1}(S_i) \underset{open}{\subset} A) \Rightarrow f : A \rightarrow B$  *continuous* (p. 103 ∈ [1])
- For *top. spaces  $(A, \mathcal{T}), (B, \mathcal{T}')$  and  $f : A \rightarrow B$* , the following are *equiv.*:
  1.  $f$  *continuous* (thm. 18.1 p. 104 ∈ [1])
  2.  $\forall U \subset A : f(\overline{U}) = \overline{f(U)}$  (thm. 18.1 p. 104 ∈ [1])
  3.  $\forall V \underset{closed}{\subset} B : f^{-1}(V) \underset{closed}{\subset} A$  (thm. 18.1 p. 104 ∈ [1])
  4.  $\forall a \in A : \forall V$  neighborhood of  $f(a) : \exists U$  neighborhood of  $a : f(U) \subset V$   
 (thm. 18.1 . 104 ∈ [1])
- Let  $f : A \rightarrow (\prod_{\alpha \in J} A_\alpha)$  be given by  $f(a) = (f_\alpha(a))_{\alpha \in J}$ , where  $f_\alpha : A \rightarrow A_\alpha$  for each  $\alpha$ . Let  $\prod A_\alpha$  have the *product topology*. Then:  
 $f$  *continuous*  $\Leftrightarrow \forall \alpha \in J : f_\alpha$  *continuous* (thm. 19.6 p. 117 ∈ [1])  
 Warning: Does *not* hold for *box topology*!

### 5.3 Constructing Continuous Functions

Premises:  $(A, \mathcal{T}), (B, \mathcal{T}'), (C, \mathcal{T}'')$  *topological spaces*

- If one of the following holds, then  $f : A \rightarrow B$  is *continuous*:
  1. *Constant function*: If  $f : A \rightarrow B$  maps *all* of  $A$  into a *single point*  $y_0 \in B$  (thm. 18.2 p. 108 ∈ [1])
  2. *Inclusion*: If  $B$  is a *subspace* of  $A$  and the *inclusion function*  $f : B \rightarrow A$  is *continuous* (thm. 18.2 p. 108 ∈ [1])
  3. *Composites*:  $g : A \rightarrow C$  *continuous*,  $h : C \rightarrow B$  *continuous* and  $f = h \circ g$  (thm. 18.2 p. 108 ∈ [1])
  4. *Restricting the domain*: If  $g : C \rightarrow B$  *continuous* and  $A$  *subspace* of  $C$  and  $f = g|_A$  (thm. 18.2 p. 108 ∈ [1])
  5. *Restricting range*: If  $g : A \rightarrow C$  *continuous* and  $B$  is a *subspace* of  $C$  *containing*  $g(A)$  and  $f$  is  $g$  with *restricted range* (thm. 18.2 p. 108 ∈ [1])
  6. *Expanding range*: If  $g : A \rightarrow C$  *continuous* and  $C$  is a *subspace* of  $B$   $f$  is  $g$  with *expanded range* (thm. 18.2 p. 108 ∈ [1])
  7. *Local continuity*: If  $A$  *can be written* as the *union of open sets*  $U_\alpha$ , such that  $\forall \alpha : f|_{U_\alpha}$  is *continuous* (thm. 18.2 p. 108 ∈ [1])
  8. *Pasting lemma*: Let  $A = S \cup T$  and  $S \cap T \neq \emptyset$ , for  $S \overset{C}{\subset} A, T \overset{C}{\subset} A$  (or  $S \overset{C}{\subset}_{open} A, T \overset{C}{\subset}_{open} A$ ). If  $g : S \rightarrow B$  *continuous*,  $h : T \rightarrow B$  *continuous* and  $\forall x \in S \cup T : g(x) = h(x)$  and  $f(x) = \begin{cases} g(x), & x \in S \\ h(x), & x \in T \end{cases}$  (thm. 18.3 p.108 ∈ [1])
- Let  $f : A \rightarrow B \times C$  be given by  $f(a) = (f_1(a), f_2(a))$ , then  $f$  *continuous*  $\Leftrightarrow f_1, f_2$  *continuous* (thm. 18.4 p. 110 ∈ [1])

### 5.4 Homeomorphisms

Premises:  $(A, \mathcal{T}), (B, \mathcal{T}')$  *topological spaces*

- $f : A \rightarrow B$  *homeomorphism*  $\stackrel{def}{=} f$  *bijective* and  $f$  and  $f^{-1}$  *both continuous* (p. 105 ∈ [1])
- $f : A \rightarrow B$  *homeomorphism*  $\equiv f$  *bijective*,  $f(U) \overset{C}{\subset}_{open} B \Leftrightarrow U \overset{C}{\subset}_{open} A$  (p. 105 ∈ [1])
- *Homeomorphisms preserve topological properties* (p. 105 ∈ [1])  
(*Homeomorphisms* is the *topology's equivalent* of *isomorphisms* in algebra)
- $f : A \rightarrow B$  is a *topological imbedding*  $\stackrel{def}{=} f : A \rightarrow B$  *injective, continuous* and  $f : A \rightarrow C$  *homeomorphism*, where  $C = f(A)$  is considered a *subspace* of  $B$  (p. 105 ∈ [1])
- Example: *Order preserving* and *bijective*  $\Rightarrow$  *homeomorphic* in the *order topology* (p. 106 ∈ [1])

## 6 Metric Topology

Premises:  $A, B$  *topological spaces*,  $(M, d)$  *metric space*,  $d$  *metric of  $M$*

- **Basis for the metric topology**  $\stackrel{def}{=}$   
 Basis  $\mathcal{B}$  = all *open spheres*  $K_d(x, r)$  for  $x \in A, r > 0$ , where  
 $K_d(x, r) = \{y \mid d(x, y) < r\}$  (p. 119 ∈ [1])  
 From this we can get the usual definition of "open" using open spheres.  
 Proving: Use sphere lemma
- For *topological space*  $A$ ,  $A$  is **metrizable**  $\stackrel{def}{=}$   
 There *exists* a *metric*  $d$  on  $A$  that *induces* the *topology* of  $A$ .  
 A **metric space** is a *metrizable space*  $A$  together with a *specific metric*  $d$   
 that gives the *topology* of  $A$ . (p. 120 ∈ [1])
- A set  $U \subset M$  is **bounded**  $\stackrel{def}{=}$   
 $\exists m \in \mathbb{R} : \forall x_1, x_2 \in U : d(x_1, x_2) < m$ . (p. 121 ∈ [1])  
*Boundedness* is *not* a *topological property*, since it depends on  $d$ .
- For  $U \subset M$ ,  $U$  *bounded* and  $U \neq \emptyset$ , the *diameter* of  $U$   $\stackrel{def}{=}$   
 $\text{diam } U = \sup\{d(x_1, x_2) \mid x_1, x_2 \in U\}$  (p. 121 ∈ [1])
- **Standard bounded metric**  $\bar{d} : M \times M \rightarrow \mathbb{R}$  *corresponding to*  $d$   $\stackrel{def}{=}$   
 $\bar{d}(x, y) = \min\{d(x, y), 1\}$ .  
 $\bar{d}$  is a *metric* that *induces* the *same topology* as  $d$ . (thm. 20.1 p. 121 ∈ [1])
- **Misc. definitions** for  $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$  (p. 121 ∈ [1])
  - **The norm** of  $\vec{x}$ :  $\|\vec{x}\| = \sqrt{x_1^2 + \dots + x_n^2}$
  - **The euclidean metric**  $d$  on  $\mathbb{R}^n$ :  $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$
  - **The square metric**  $\rho$  on  $\mathbb{R}^n$ :  $\rho(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$
- Let  $d$  and  $d'$  be *two metrics* on  $A$ , and let  $\mathcal{T}$  and  $\mathcal{T}'$  be the *topologies* they *induce*.  
 $\mathcal{T}'$  *finer than*  $\mathcal{T}$  (i.e.  $\mathcal{T} \subset \mathcal{T}'$ )  $\Leftrightarrow$   
 $\forall x \in A : \forall \epsilon > 0 : \exists \delta > 0 : K_{d'}(x, \delta) \subset K_d(x, \epsilon)$  (lemma 20.2 p. 122 ∈ [1])
- The *topologies* on  $\mathbb{R}^n$  *induced* by the *euclidean metric*  $d$  and the *square metric*  $\rho$  are the *same* as the *product topology* on  $\mathbb{R}^n$ . (thm. 20.3 p. 123 ∈ [1])
- In  $\mathbb{R}^\omega$ ,  $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$  and  $\rho(\vec{x}, \vec{y}) = \sup\{|x_n - y_n|\}$  does *not always converge*, and does thus *not always* define a *metric* of  $\mathbb{R}^\omega$ . (p. 124 ∈ [1])
- **The uniform metric**  $\bar{\rho}$  on  $\mathbb{R}^J$  for *arbitrary index set*  $J$   $\stackrel{def}{=}$   
 $\bar{\rho}(\vec{x}, \vec{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$ , where  $\bar{d}$  is *standard bounded metric* on  $\mathbb{R}^J$  and  $\vec{x} = (x_\alpha)_{\alpha \in J}$ ,  $\vec{y} = (y_\alpha)_{\alpha \in J}$ . (p. 124 ∈ [1])
- **The uniform topology** on  $\mathbb{R}^J$  for *arbitrary index set*  $J$   $\stackrel{def}{=}$   
 The *topology induced* by the *uniform metric* on  $\mathbb{R}^J$ . (p. 124 ∈ [1])
- **We define**  $D(\vec{x}, \vec{y}) = \sup\{\frac{\bar{d}(x_i, y_i)}{i}\}$  **on**  $\mathbb{R}^\omega$  **for**  $\vec{x}, \vec{y} \in \mathbb{R}^\omega$ ,  
 where  $\bar{d}$  is the *standard bounded metric*.  
 $D$  *induces* the *product topology* on  $\mathbb{R}^\omega$  (thm. 20.5 p. 125 ∈ [1])

- *Metrics vs. topologies:* (p. 129 ∈ [1])
  - $B \stackrel{\subset}{\text{subspace}} A$ ,  $d$  is a *metric* for the *topology* on  $A \Rightarrow d|_{B \times B}$  is a *metric* for the *topology* on  $B$
  - *Some order topologies* are *metrizable* (e.g.  $\mathbb{Z}_+, \mathbb{R}$ ) and *others are not*
  - *Hausdorff axiom* holds for every *metric topology*
  - *Countable products* of *metrizable spaces* are *metrizable*
- Let  $f : A \rightarrow B$  and  $A, B$  be *metrizable topological spaces* with *metrics*  $d_A, d_B$  respectively, then:  $f$  *continuous*  $\Leftrightarrow \forall x \in A : \forall \epsilon > 0 : \exists \delta > 0 : (d_A(x, y) < \delta) \Rightarrow (d_B(f(x), f(y)) < \epsilon)$  (thm. 21.1 p. 129 ∈ [1])
- Let  $U \subset A$ . There is a *sequence of points* of  $U$  converging to  $x \Rightarrow x \in \overline{U}$ . The *converse* hold if  $A$  is *metrizable* (or just satisfies the *first countability axiom*). Called the *sequence lemma*. (lemma 21.2 p. 130 ∈ [1])
- Let  $f : A \rightarrow B$ . If  $f$  is *continuous*, then for *every convergent sequence*  $x_n \rightarrow x$ ,  $f(x_n)$  *converges* to  $f(x)$ . The *converse* hold if  $A$  is *metrizable* (or just satisfies the *first countability axiom*). (lemma 21.3 p. 130 ∈ [1])
- The *topological space*  $A$  has a *countable basis* at  $x \in A \stackrel{\text{def}}{=} \text{There exists a countable collection of neighborhoods of } x, \text{ such that any neighborhood } U \text{ of } x \text{ contains at least one of the } U_n.$  (p. 130 ∈ [1])
- The *topological space*  $A$  satisfies the *first countability axiom*  $\stackrel{\text{def}}{=} \forall x \in A : x \text{ has a countable basis}$  (p. 131 ∈ [1])
- Let  $U$  be a *set* and  $f_n : U \rightarrow M$  a *sequence of functions*, then:  $f_n$  *converges uniformly to*  $f : U \rightarrow M \stackrel{\text{def}}{=} \forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n > N : \forall x \in U : d(f_n(x), f(x)) < \epsilon$  (p. 131 ∈ [1])
- *Uniform limit theorem:* Let  $U$  be a *set* and  $f_n : U \rightarrow M$  a *sequence of continuous functions*. If  $(f_n)$  *converges uniformly to*  $f$ , then  $f$  is *continuous* (thm. 21.6 p. 132 ∈ [1])
- In the space  $\mathbb{R}^X$  of *functions*  $f : X \rightarrow \mathbb{R}$  with the *uniform metric*  $\bar{\rho}$ : A *sequence*  $(f_n)$  *converges uniformly to*  $f \Leftrightarrow f_n$  *converges to*  $f$  w.r.t.  $\bar{\rho}$ . (p. 132 ∈ [1])

## 6.1 Well-Known Facts

- The *discrete topology* is *induced* as a *metric topology* by the *discrete metric*.  
(ex. 1 p. 120 ∈ [1])
- The *order topology* on  $\mathbb{R}$  is *induced* as a *metric topology* by the *standard metric* on  $\mathbb{R} : d(x, y) = |x - y|$ . (ex. 2 p. 120 ∈ [1])
- *Non-metrizable spaces*:
  - $\mathbb{R}^J$ , where  $J$  is *uncountable* with *product topology*, *box topology* or *uniform topology* (p. 125 ∈ [1])
  - $\mathbb{R}^J$ , where  $J$  is *countable* with *box topology* or *uniform topology* (p. 125 ∈ [1])
- *Continuous functions*, where  $f, g$  are *continuous*:
  - $(f + g), (f - g)$  and  $(f \cdot g)$  are *continuous*
  - If  $\forall x : g(x) \neq 0$ , then  $(f/g)$  is *continuous*
  - On  $\mathbb{R} \times \mathbb{R} : +, -, \cdot$  *operations* (lemma 21.4 p. 131 ∈ [1])
  - On  $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) : /$ , the *divide operation* (lemma 21.4 p. 131 ∈ [1])

## 7 Quotient Topology

Premises:  $A, B$  *topological spaces*

- Let  $p : A \rightarrow B$  be *surjective*.  
 $p$  is a **quotient map** / has **strong continuity**  $\stackrel{def}{=}$   
 $\forall U \subset B : (U \text{ open} \Leftrightarrow p^{-1}(U) \text{ open}) \equiv$   
 $\forall U \subset B : (U \text{ closed} \Leftrightarrow p^{-1}(U) \text{ closed})$  (p. 137 ∈ [1])
- $U \subset A$  is a **saturated subset** of  $A$  w.r.t. *surjective map*  $p : A \rightarrow B \stackrel{def}{=}$   
 $\forall y \in A : (p^{-1}(\{y\}) \text{ intersects } U \Rightarrow p^{-1}(\{y\}) \subset U)$  (p. 137 ∈ [1])  
 Thus  $U$  is *saturated* if it equals the *complete inverse image* of a *subset* of  $B$ .
- Let  $p : A \rightarrow B$  be *surjective*.  $p : A \rightarrow B$  is a **quotient map**  $\equiv$  (p. 137 ∈ [1])  
 $p$  *continuous* and maps *saturated open sets* of  $A$  into *open sets* of  $B \equiv$   
 $p$  *continuous* and maps *saturated closed sets* of  $A$  into *closed sets* of  $B$
- **Open quotient map**  $p : A \rightarrow B \stackrel{def}{=} \equiv$  (p. 137 ∈ [1])  
 $p$  *quotient map* and  $p$  *open map* (I.e.:  $\forall U \stackrel{C}{\text{open}} A : p(U) \text{ open in } B$ )
- **Closed quotient map**  $p : A \rightarrow B \stackrel{def}{=} \equiv$  (p. 137 ∈ [1])  
 $p$  *quotient map* and  $p$  *closed map* (I.e.:  $\forall U \stackrel{C}{\text{closed}} A : p(U) \text{ closed in } B$ )
- $p : A \rightarrow B$  *continuous, surjective* and *either open or closed*  $\Rightarrow$   
 $p$  *quotient map*. (p. 137 ∈ [1])  
 There exists quotient maps that are *neither open nor closed*.
- Let  $X$  be a *set* and  $p : A \rightarrow X$  a *surjective map*.  
**The quotient topology**  $\mathcal{T}$  (which is *unique!*) **on**  $X$  **induced by**  $p \stackrel{def}{=}$   
 The *topology*  $\mathcal{T}$  on  $X$  relative to which  $p$  is a *quotient map*. (p. 138 ∈ [1])  
 $\mathcal{T}$  is *defined* as the *subsets*  $U$  of  $X$ , such that  $p^{-1}(U)$  are *open* in  $A$ .
- Let  $A^*$  be a *partition* of  $A$  into *disjoint subsets* whose union is  $A$ .  
 Let  $p : A \rightarrow A^*$  be the *surjective map* that *carries each point* of  $A$  to the *element* of  $A^*$  that *contains* it. In the *quotient topology induced* by  $p$ ,  $A^*$  is called the **quotient space of**  $A$ . (p. 179 ∈ [1])  
 Often called an **identification space of**  $A$  or a **decomposition space of**  $A$ .

## 7.1 Quotient maps vs. quotient space

Premises:  $A, B$  *topological spaces*

- Warning: If  $p : A \rightarrow B$  is a *quotient map* and  $C \stackrel{\subset}{\text{subspace}} A$ , then  $q : C \rightarrow p(C)$  obtained by *restricting*  $p$  need not be a *quotient map*! (p. 140 ∈ [1])
- Let  $p : A \rightarrow B$  be a *quotient map* and  $C \stackrel{\subset}{\text{subspace}} A$ , where  $C$  is *saturated* w.r.t.  $p$ . Let  $q : C \rightarrow p(C)$  be  $p$  restricted to  $C$ , then  $q$  is a *quotient map* if one of the following holds:
  1. If  $C$  is either *open* or *closed*. (thm. 22.1 p. 140 ∈ [1])
  2. If  $p$  is either an *open map* or a *closed map*. (thm. 22.1 p. 140 ∈ [1])
- $p, q$  *quotient maps*  $\Rightarrow p \circ q$  *quotient map* (p. 141 ∈ [1])
- If  $p, q$  are *quotient maps*, then  $(p, q)$  is a *quotient map* if one of the following holds (but not in general!):
  1. If the *spaces* have *local compactness* (p. 141 ∈ [1])
  2. If  $p$  and  $q$  are *open maps* (p. 141 ∈ [1])
- If  $A$  is a *Hausdorff space*, then:
  1. If each element of  $A^*$  is a *closed subset* of  $A$ , then the *quotient space*  $A^*$  satisfies the  $T_1$ -*axiom* (p. 141 ∈ [1])
  2. For  $A^*$  to be *Hausdorff*, it's *harder* to find conditions (p. 141 ∈ [1])
- Let  $A, B, C$  be *topological spaces* and  $p : A \rightarrow B$  be a *quotient map*. Let  $g : A \rightarrow C$  be a map which is *constant* on each set  $p^{-1}(\{b\})$ , for  $b \in B$ . Then  $g$  induces the map  $f : B \rightarrow C$ , such that  $f \circ p = g$ , and we have:
  1.  $f$  *continuous*  $\Leftrightarrow g$  *continuous* (thm. 22.2 p. 142 ∈ [1])
  2.  $f$  *quotient*  $\Leftrightarrow g$  *quotient* (thm. 22.2 p. 142 ∈ [1])
- Let  $g : A \rightarrow B$  be a *surjective continuous map*. Let  $A^*$  be the following *collection* of subsets of  $A$ :  $A^* = \{g^{-1}(\{b\}) \mid b \in B\}$ . Give  $A^*$  the *quotient topology*, then we have:
  1.  $g$  induces a *bijective continuous map*  $f : A^* \rightarrow B$ , which is a *homeomorphism* if and only if  $g$  is a *quotient map*. (cor. 22.3 p. 142 ∈ [1])
  2.  $B$  *Hausdorff*  $\Rightarrow A^*$  *Hausdorff* (cor. 22.3 p. 142 ∈ [1])
- Warning: The *product* of 2 *quotient maps* need not be a *quotient map* (ex. 7 p. 143 ∈ [1])
- A **retraction** of  $A$  into  $U$ , where  $U \subset A \stackrel{\text{def}}{=} a$  a *continuous map*  $r : A \rightarrow U$ , where  $\forall a \in U : r(a) = a$  (exc. 22;2 p. 144 ∈ [1])
- $f$  is a *retraction*  $\Rightarrow f$  is a *quotient map* (exc. 22;2 p. 144 ∈ [1])

## 8 Connectedness and Compactness

### 8.1 Fundamental Calculus Theorems depending on Compactness and Connectedness

- *Intermediate value theorem:* If  $f : [a, b] \rightarrow \mathbb{R}$  *continuous* and  $r \in \mathbb{R}$  and  $f(a) < r < f(b)$ , then  $\exists c \in [a, b] : f(c) = r$ .  
Depends on the *connectedness* of  $[a, b]$ . (p. 147 ∈ [1])
- *Maximum value theorem:* If  $f : [a, b] \rightarrow \mathbb{R}$  *continuous*, then  $\exists c \in [a, b] : \forall x \in [a, b] : f(x) \leq f(c)$ .  
Depends on the *compactness* of  $[a, b]$ . (p. 147 ∈ [1])
- *Uniform continuity theorem:* If  $f : [a, b] \rightarrow \mathbb{R}$  *continuous*, then  $\forall \epsilon > 0 : \exists \delta > 0 : \forall x_1, x_2 \in [a, b] : |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$ .  
Depends on the *compactness* of  $[a, b]$ . (p. 147 ∈ [1])

## 8.2 Connected Spaces

Premises:  $A, B, C$  *topological spaces*

- **A separation of  $A$**   $\stackrel{def}{=} U, V \overset{\subset}{\underset{open}{\subset}} A$ , where  $U, V \neq \emptyset$  and  $U \cap V = \emptyset$  and  $U \cup V = A$ . (p. 148 ∈ [1])
- $A$  is **connected**  $\stackrel{def}{=} \text{there does not exist any separation of } A$ . (p. 148 ∈ [1])
- $A$  is **connected**  $\Leftrightarrow$  the *only subsets* of  $A$  that are *both open and closed* are  $\emptyset$  and  $A$ . (p. 148 ∈ [1])
- For  $B \overset{\subset}{\underset{subspace}{\subset}} A$ , a **separation of  $B$**   $\stackrel{def}{=} U, V \overset{\subset}{\underset{subset}{\subset}} B$ , where  $U, V$  *nonempty and disjoint* and  $U \cup V = B$  and *neither  $U$  nor  $V$  contains a limit point of the other set*. (thm. 23.1 p. 148 ∈ [1])
- For  $B \overset{\subset}{\underset{subspace}{\subset}} A$ ,  $B$  is **connected**  $\stackrel{def}{=} \text{there does not exist any separation of } B$ . (thm 23.1 p. 148 ∈ [1])
- Examples of *separations*:
  1.  $] - 1, 0[ \cup ] 0, 1[$  as a *subspace* of  $\mathbb{R}$ . (ex. 2 p. 149 ∈ [1])
  2. Any *subspace* of  $\mathbb{Q}$  bigger than a *one-element set*. (ex. 4 p. 149 ∈ [1])
- Examples of *connected spaces*:
  1. A *2-point set* with *indiscrete topology* (ex. 1. p. 149 ∈ [1])
  2.  $[-1, 1]$  (ex. 3. p. 149 ∈ [1])
- If  $U, V$  form a *separation* of  $A$  and  $B$  is a *connected subspace* of  $A$ , then:  $B$  lies *entirely within either  $U$  or  $V$*  (lemma 23.2 p. 149 ∈ [1])
- A *collection  $A_\alpha$  of connected subspaces* of  $A$  with a *point in common*  $\Rightarrow \bigcup A_\alpha$  is *connected* (thm. 23.3 p. 159 ∈ [1])
- Let  $B$  be a *connected subspace* of  $A$ . If  $B \subset C \subset \overline{B}$ , then  $C$  is *also connected*. (thm. 23.4 p. 150 ∈ [1])  
I.e. if  $C$  is formed by *adding some or all of  $B$ 's limit points* to  $B$ , then  $C$  is *connected*.
- The *image* of a *connected space* under *continuous map* is *connected*. (thm. 23.5 p. 150 ∈ [1])
- *Finite cartesian products of connected spaces* are *connected*. (thm. 23.6 p. 150 ∈ [1])  
E.g.: 1)  $\mathbb{R}^\omega$  is *not connected* in the *box topology*. (ex. 6 p. 151 ∈ [1])  
2)  $\mathbb{R}^\omega$  is *connected* in the *product topology*. (ex. 7 p. 151 ∈ [1])
- For an *arbitrarily indexed family*  $\{A_\alpha\}_{\alpha \in J}$  of *connected spaces*, the *product space*  $\prod_{\alpha \in J} A_\alpha$  is *connected*. (exc. 23;10 p. 152 ∈ [1])
- A *space  $A$  is totally disconnected*  $\stackrel{def}{=} \text{the only connected subspaces are one-point sets}$ . (exc. 23;5 p. 152 ∈ [1])

### 8.3 Connected Subspaces of the Real Line

- **A linear continuum**  $L \stackrel{def}{=} \langle p. 153 \in [1] \rangle$   
a set  $L$  with the ordering properties of  $\mathbb{R}$ . I.e.:
  1.  $L$  is simply ordered
  2.  $L$  has more than one element
  3.  $L$  has the least upper bound property
  4.  $\forall x, y \in L : x < y \Rightarrow \exists z \in L : x < z < y$
- If  $L$  is a linear continuum in the order topology, then  $L$  is connected and so are rays and intervals in  $L$ .  $\langle thm. 24.1 p. 153 \in [1] \rangle$
- For  $B \stackrel{\subset}{subspace} A$ ,  $B$  is a **convex subspace of  $A$**   $\stackrel{def}{=} \forall a, b \in B, a < b : [a, b] \subset B$ .  $\langle p. 153 \in [1] \rangle$
- $\mathbb{R}$  is connected and so are intervals and rays in  $\mathbb{R}$ .  $\langle cor. 24.2 p. 154 \in [1] \rangle$
- **Intermediate value theorem:** Let  $f : A \rightarrow B$  be a continuous map, where  $A$  is connected and  $B$  is an ordered set in the order topology. If  $a, b \in A, r \in B$  and  $f(a) < r < f(b)$ , then  $\exists c \in A : f(c) = r$ .  $\langle thm. 24.3 p. 154 \in [1] \rangle$
- Examples of linear continua:  $\langle ex. 1, 2 p. 155 \in [1] \rangle$ 
  1. The ordered square is a linear continuum.
  2. If  $A$  is a well-ordered set, then:  $A \times [0, 1[$  is a linear continuum.
- **A path in  $A$  from  $x \in A$  to  $y \in A$**   $\stackrel{def}{=} \text{a continuous map } f : [a, b] \rightarrow A$ , where  $a, b \in \mathbb{R}, [a, b] \subset \mathbb{R}, f(a) = x, f(b) = y$ .  $\langle p. 155 \in [1] \rangle$
- **A space  $A$  is path connected**  $\stackrel{def}{=} \forall a, b \in A : \text{there exists a path in } A \text{ from } a \text{ to } b$ .  $\langle p. 155 \in [1] \rangle$
- A space  $A$  is path connected  $\Rightarrow A$  is connected (but the converse does not hold in general).  $\langle p. 155 \in [1] \rangle$
- The image of a path connected space under a continuous map is path connected.  $\langle ex. 5 p. 156 \in [1] \rangle$
- Examples of path connected spaces:
  1. Open and closed balls in  $\mathbb{R}^n$  are path connected.  $\langle ex. 3 p. 156 \in [1] \rangle$
  2. The punctured euclidean plane  $\mathbb{R}^n \setminus \{\vec{0}\}$  is path connected for  $n > 1$ .  $\langle ex. p. 156 \in [1] \rangle$
  3. The unit sphere surface  $S^{n-1}$  in  $\mathbb{R}^n$  defined by  $S^{n-1} = \{\vec{x} \mid \|\vec{x}\| = 1\}$  for  $n > 1$  is path connected.  $\langle ex. 5 p. 156 \in [1] \rangle$
- Examples of non-path connected spaces:
  1. The ordered square  $I_o^2$  is connected but not path connected.  $\langle ex. 6 p. 156 \in [1] \rangle$
  2. The closure  $\bar{S}$  of the Topologist's Sine Curve  $S = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}$  is connected but not path connected.  $\langle ex. 7 p. 156-157 \in [1] \rangle$

## 8.4 Components and Local Connectedness

- The **components** / **connected components** of a top. space  $A \stackrel{def}{=} \underline{\underline{\quad}}$  the equivalence classes of the equivalence relation  $\sim$  where  $x \sim y$  means that there exists a connected subspace of  $A$  containing  $x$  and  $y$ . (p. 159 ∈ [1])
- The **components** of a topological space  $A \equiv$  **connected disjoint subspaces** of  $A$  whose **union** is  $A$ , such that **each non-empty connected subspace** of  $A$  intersects **only one** of them. (thm. 25.1 p. 159 ∈ [1])
- The **path components** of a topological space  $A \stackrel{def}{=} \underline{\underline{\quad}}$  the equivalence classes of the equivalence relation  $\sim$  where  $x \sim y$  means that there exists a **path** in  $A$  from  $x$  to  $y$ . (p. 160 ∈ [1])
- The **path components** of a topological space  $A \equiv$  **path connected disjoint subspaces** of  $A$  whose **union** is  $A$ , such that **each non-empty path connected subspace** of  $A$  intersects **only one** of them. (thm. 25.2 p. 160 ∈ [1])
- Each **component** is **closed** in a topological space  $A$ . (p. 160 ∈ [1])
- If a topological space  $A$  only has **finitely many components**, then **each component** is **open** in  $A$ . (p. 160 ∈ [1])
- Warning: **Path components** need **neither be open nor closed**. (p. 160 ∈ [1])
- A topological space  $A$  is **locally connected at**  $x \in A \stackrel{def}{=} \underline{\underline{\quad}}$  for every neighborhood  $U$  of  $x$ , there is a **connected neighborhood**  $V$  of  $x$ , where  $V \subset U$ . (p. 161 ∈ [1])
- A topological space is **locally connected**  $\stackrel{def}{=} \underline{\underline{\quad}}$   $\forall x \in A : A$  is **locally connected at**  $x$ . (p. 161 ∈ [1])
- A topological space  $A$  is **locally path connected at**  $x \in A \stackrel{def}{=} \underline{\underline{\quad}}$  for every neighborhood  $U$  of  $x$ , there is a **path connected neighborhood**  $V$  of  $x$ , where  $V \subset U$ . (p. 161 ∈ [1])
- A topological space is **locally path connected**  $\stackrel{def}{=} \underline{\underline{\quad}}$   $\forall x \in A : A$  is **locally path connected at**  $x$ . (p. 161 ∈ [1])
- A topological space  $A$  is **locally connected**  $\Leftrightarrow$  (thm. 25.3 p. 161 ∈ [1]) for every open set  $U$  of  $A$ , each **component** of  $U$  is **open** in  $A$ .
- A topological space  $A$  is **locally path connected**  $\Leftrightarrow$  (thm. 25.4 p. 161 ∈ [1]) for every open set  $U$  of  $A$ , each **path component** of  $U$  is **open** in  $A$ .
- For topological space  $A$ :  
Each **path component** of  $A$  lies in a **component** of  $A$ . (thm. 25.5 p. 161 ∈ [1])
- For topological space  $A$ : If  $A$  is **locally path connected**, then the **components** and the **path components** of  $A$  are the **same**. (thm. 25.5 p. 161 ∈ [1])

## 9 Compact Spaces

Premises:  $A, B$  *topological spaces*

- A *collection*  $\mathcal{A}$  of subsets of  $A$  covers  $A$  / is a *covering* of  $A \stackrel{def}{=} \bigcup \mathcal{A}$   
the union of the elements of  $\mathcal{A}$  is equal to  $A$  (i.e.  $\bigcup \mathcal{A} = A$ ). (p. 164 ∈ [1])
- A *collection*  $\mathcal{A}$  of subsets of  $A$  is an *open covering* of  $A \stackrel{def}{=} \bigcup \mathcal{A}$   
 $\mathcal{A}$  covers  $A$  and  $\forall A_i \in \mathcal{A} : A_i \stackrel{C}{\underset{open}{\subset}} A$ . (p. 164 ∈ [1])
- $A$  is *compact*  $\stackrel{def}{=} \langle p. 164 \in [1] \rangle$   
every open covering  $\mathcal{A}$  of  $A$  has a finite subcollection that also covers  $A$ .
- For  $B \stackrel{C}{\underset{subspace}{\subset}} A$ : A *collection*  $\mathcal{A}$  of subsets of  $A$  covers  $B \stackrel{def}{=} B \subset \bigcup \mathcal{A}$ . (p. 164 ∈ [1])
- For  $B \stackrel{C}{\underset{subspace}{\subset}} A$ :  $B$  is *compact*  $\Leftrightarrow$  every covering of  $B$  by sets open in  $A$  has a finite subcollection covering  $B$ . (lemma 26.1 p. 164 ∈ [1])
- For  $B \stackrel{C}{\underset{subspace}{\subset}} A$ : If  $A$  compact and  $B \stackrel{C}{\underset{closed\ subspace}{\subset}} A$ , then  $B$  compact. (thm. 26.2 p. 165 ∈ [1])
- For  $B \stackrel{C}{\underset{subspace}{\subset}} A$ : If  $A$  Hausdorff and  $B$  compact, then  $B$  closed. (thm. 26.3 p. 165 ∈ [1])
- For  $B \stackrel{C}{\underset{subspace}{\subset}} A$ : If  $A$  Hausdorff and  $B$  compact and  $x_0 \notin B$ , then there exist disjoint open sets  $U$  and  $V$  of  $A$  containing  $x_0$  and  $B$  respectively. (lemma 26.4 p. 166 ∈ [1])

## 9.1 Compact Spaces as Tables

Premises:  $A, B$  *topological spaces*,  $\mathcal{A}$  is a *collection of subsets of A*.

$A$	$B$	property		equivalent
		$\mathcal{A}$ covers $A$ a.k.a. $\mathcal{A}$ is a <i>covering</i> of $A$	$\stackrel{def}{=}$	$\cup \mathcal{A} = A$ (p. 164 ∈ [1])
		$\mathcal{A}$ is a <i>open covering</i> of $A$	$\stackrel{def}{=}$	$\mathcal{A}$ covers $A$ and <i>all elements in <math>\mathcal{A}</math> are open</i> (p. 164 ∈ [1])
	$B \overset{\subset}{\text{subspace}} A$	$\mathcal{A}$ covers $B$	$\stackrel{def}{=}$	$B \subset \cup \mathcal{A}$ (p. 164 ∈ [1])
		$A$ <i>compact</i>	$\stackrel{def}{=}$	<i>every open covering <math>\mathcal{A}</math> of <math>A</math> has a <i>finite subcollection</i> that <i>also covers <math>A</math></i></i> (p. 164 ∈ [1])
<i>compact</i>	$B \overset{\subset}{\text{subspace}} A$		$\Leftrightarrow$	<i>every covering of <math>B</math> by <i>sets open in <math>A</math></i> has a <i>finite subcollection covering <math>B</math></i></i> (lemma 26.1 p. 164 ∈ [1])

Premises:  $A, B$  *topological spaces*,  $B \overset{\subset}{\text{subspace}} A$ .

$B$	$\overset{\subset}{\text{subspace}}$	$A$	Misc.	$\Rightarrow$
	$\overset{\subset}{\text{closed subspace}}$	<i>compact</i>		$B$ <i>compact</i> (thm. 26.2 p. 165 ∈ [1])
<i>compact</i>		<i>Hausdorff</i>		$B$ <i>closed</i> (thm. 26.3 p. 165 ∈ [1])
<i>compact</i>		<i>Hausdorff</i>	$x_0 \notin B$	there exist <i>disjoint open sets</i> $U$ and $V$ of $A$ <i>containing <math>x_0</math> and <math>B</math> respectively</i> (thm. 26.4 p. 165 ∈ [1])

## 9.2 Compact Spaces Continued

Premises:  $A, B$  *topological spaces*

- If  $A$  *compact* and  $f$  *continuous*, then  $f(A)$  *compact*. (thm. 26.5 p. 166 ∈ [1])
- If  $A$  *compact*,  $B$  *Hausdorff* and  $f : A \rightarrow B$  *bijective* and *continuous*, then  $f$  *homeomorphism*. (thm. 26.6 p. 167 ∈ [1])
- *Tychonoff theorem*:  
For  $\{A_i\}_{i \in J}$  for *arbitrary* index set  $J$ , where  $\forall i \in J : A_i$  *compact*:  
 $\prod_{i \in J} A_i$  *compact*. (thm. 26.7 p. 167, thm. 37.3 p. 224 ∈ [1])
- The *tube lemma*: For *product space*  $A \times B$ , where  $B$  *compact*:  
If  $N \stackrel{c}{\subset} A \times B$ , where  $x_0 \in A$  and the *slice*  $x_0 \times B \subset N$ , then  $N$  *contains some tube*  $W \times B$  about  $x_0 \times B$ ,  
where  $W$  is a *neighborhood* of  $x_0$  in  $A$ . (lemma 26.8 p. 168 ∈ [1])
- *Warning*: The *tube lemma* does *not hold* if  $B$  is *not compact*!  
Example:  $B$  the  $y$ -axis of  $\mathbb{R}^2$  and  $N = \{(x, y) \mid |x| < \frac{1}{y^2+1}\}$  and  $x_0 = 0$ .  
Then  $N$  is *open*,  $0 \times \mathbb{R} \subset N$ , but  $N$  does *not contain any tube* about  $0 \times \mathbb{R}$ .  
(ex. 7 p. 168 ∈ [1])
- The *collection*  $\mathcal{C}$  of *subsets* of the *topological space*  $A$  has  
**the *finite intersection property***  $\stackrel{def}{=} (p. 169 \in [1])$   
for *every finite subcollection*  $\{C_1, \dots, C_n\}$  of  $\mathcal{C}$ :  $(C_1 \cap \dots \cap C_n) \neq \emptyset$ .
- For *topological space*  $A$ :  $A$  *compact*  $\Leftrightarrow$   
for *every subcollection*  $\mathcal{C}$  of *closed sets* in  $A$  having the *finite intersection property*:  $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$  (i.e. the *intersection* of *all*  $C \in \mathcal{C}$  is *non-empty*).  
(thm. 26.9 p. 169 ∈ [1])
- For *compact topological space*  $A$ :  
A *nested sequence*  $C_1 \supset C_2 \supset \dots \supset C_n \supset C_{n+1} \supset \dots$  of *closed sets* in  $A$ ,  
where each  $C_n \neq \emptyset$  automatically has the *finite intersection property*,  
so  $\bigcap_{n \in \mathbb{Z}_+} C_n \neq \emptyset$ . (p. 170 ∈ [1])

## 9.3 Basic Metric Stuff (Distance and Diameter)

- For  $(M, d)$  *metric space*,  $C \subset M$ ,  $C \neq \emptyset$ :  
For each  $x \in M$ , **the *distance from*  $x$  to  $A$**   $\stackrel{def}{=}$   
 $d(x, A) = \inf\{d(x, a) \mid a \in A\}$ .  
(p. 175 ∈ [1])
- $d(x, A)$  is a *continuous function* of  $x$ . (p. 175 ∈ [1])
- For  $(M, d)$  *metric space*:  
**The *diameter*  $diam(C)$  of a *bounded subset*  $C$  of  $M$**   $\stackrel{def}{=}$   
 $diam(C) = \sup\{d(x_1, x_2) \mid x_1, x_2 \in C\}$ .  
(p. 175 ∈ [1])

## 9.4 Compact Subspaces of the Real Line

- For *simply ordered set*  $A$  having the *least upper bound property*:  
In the *order topology*, each *closed interval* in  $A$  is *compact*.  
(thm. 27.1 p. 127 ∈ [1])
- *Every closed interval* in  $\mathbb{R}$  is *compact*. (cor. 27.2 p. 173 ∈ [1])
- For  $A \stackrel{C}{\text{subspace}} \mathbb{R}^2$ :  $A$  *compact*  $\Leftrightarrow$   
 $A$  *closed and bounded* in the *euclidean metric*  $d$  or the *square metric*  $\rho$ .  
(thm. 27.3 p. 173 ∈ [1])
- *Extreme Value Theorem*:  
For  $B$  *ordered set* in *order topology* and  $f : A \rightarrow B$  *continuous*:  
If  $A$  *compact*, then there *exists points*  $c, d \in A$  such that  
 $\forall x \in B : f(c) \leq f(x) \leq f(d)$ .  
I.e.:  $A$  *compact*  $\Rightarrow \exists c, d \in A : \forall x \in B : f(c) \leq f(x) \leq f(d)$ .  
(thm. 27.4 p. 174 ∈ [1])
- The *Lebesgue Number Lemma*:  
Let  $\mathcal{A}$  be an *open covering* of the *metric space*  $(M, d)$ .  
If  $M$  is *compact*, there is a  $\delta > 0$  such that for *each subset*  $C$  of  $M$  having  
a *diameter less than*  $\delta$ , there *exists an element of*  $\mathcal{A}$  *containing*  $C$ .  
I.e.:  $M$  *compact*  $\Rightarrow \exists \delta > 0 : \forall C \subset M, \text{diam}(C) < \delta : \exists A \in \mathcal{A} : C \subset A$ .  
 $\delta$  is a *Lebesgue number* for the *covering*  $\mathcal{A}$ . (lemma 27.5 p. 175 ∈ [1])
- For *metric spaces*  $(A, d_A), (B, d_B)$ :  
 $f : A \rightarrow B$  is **uniformly continuous**  $\stackrel{\text{def}}{=} \forall \epsilon > 0 : \exists \delta > 0 : \forall x_1, x_2 \in M : d_A(x_1, x_2) < \delta \Rightarrow d_B(f(x_1), f(x_2)) < \epsilon$ .  
(p. 176 ∈ [1])
- *Uniform continuity theorem*: For  $(A, d_A), (B, d_B)$  *metric spaces*:  
 $f : A \rightarrow B$  *continuous* and  $A$  *compact*  $\Rightarrow f$  *uniformly continuous*.  
(thm. 27.6 p. 176 ∈ [1])
- For *topological space*  $A$ :  $x \in A$  is an **isolated point of**  $A \stackrel{\text{def}}{=} \{x\}$  *is open* in  $A$ . (p. 176 ∈ [1])
- For *compact Hausdorff space*  $A \neq \emptyset$ :  
If  $A$  has *no isolated points*, then  $A$  is *uncountable*. (thm. 27.7 p. 176 ∈ [1])
- $\mathbb{R}$  is *uncountable*. (p. 176-177 ∈ [1])
- *Every closed interval* of  $\mathbb{R}$  is *uncountable*. (cor. 27.8 p. 177 ∈ [1])

## 9.5 Limit Point Compactness

- For *topological space*  $A$ :  $A$  is **limit point compact** a.k.a. **Fréchet compact** a.k.a. **has the Bolzano-Weierstrass property**  $\stackrel{def}{=}$  every infinite subset of  $A$  has a *limit point*. (p. 178 ∈ [1])
- For *topological space*  $A$ :  $A$  compact  $\Rightarrow$   $A$  *limit point compact* (but not conversely) (thm. 28.1 p. 179 ∈ [1])
- For set  $C$ ,  $(x_n)$  a *sequence of points* in  $C$ :  
 $(x_{n_i})$  is a **subsequence of**  $(x_n)$   $\stackrel{def}{=}$   $n_1 \leq n_2 \leq \dots n_i \leq \dots$  is an *increasing sequence of integers*. (p. 179 ∈ [1])
- For *topological space*  $A$ :  $A$  is **sequentially compact**  $\stackrel{def}{=}$  every *sequence of points* of  $A$  has a *convergent subsequence*. (p. 179 ∈ [1])
- For *metrizable topological space*  $A$ , the following are equivalent:
  - $A$  is *compact*
  - $A$  is *limit point compact*
  - $A$  is *sequentially compact*

(thm. 28.2 p. 179 ∈ [1])

## 9.6 Local Compactness

- For *topological space*  $A$ :  $A$  is **locally compact at**  $x \in A \stackrel{def}{=} \text{there exists a } C \underset{\text{compact subspace}}{\subset} A$ , where  $C$  contains a *neighborhood* of  $x$ .  
(p. 182 ∈ [1])
- For *topological space*  $A$ :  $A$  is **locally compact**  $\stackrel{def}{=} \forall x \in A : A$  is locally compact at  $x$ . (p. 182 ∈ [1])
- For *topological space*  $A$ :  $A$  compact  $\Rightarrow A$  locally compact. (p. 182 ∈ [1])
- $\mathbb{R}^n$  is *locally compact*, but  $\mathbb{R}^\omega$  is *not*. (ex. 2 p. 182-183 ∈ [1])
- Every *simply ordered set*  $A$  with the *least upper bound property* is *locally compact*. (ex. 3 p. 183 ∈ [1])
- For *topological space*  $A$ :  $A$  is *locally compact* and *Hausdorff*  $\Leftrightarrow$  there exists a *topological space*  $B$  where:
  1.  $A \underset{\text{subspace}}{\subset} B$ .
  2.  $B \setminus A$  consist of a *single point*.
  3.  $B$  is a *compact Hausdorff space*.

If  $B$  and  $B'$  satisfies these properties, then there is a *homeomorphism* of  $B$  with  $B'$  that *equals the identity* map on  $A$ . (thm. 29.1 p. 183 ∈ [1])  
 If  $A$  is *compact*: Then  $B$  is  $A$  with a *single isolated point added*.  
 If  $A$  is *not compact*: Then  $B \setminus A$  is a *limit point* of  $A$  so that  $\bar{A} = B$ .  
 (p. 184 ∈ [1])
- For *topological spaces*  $A, B$ :  $B$  is a **compactification of**  $A \stackrel{def}{=} B$  is a *compact Hausdorff space* and  $A \underset{\text{subspace}}{\subset} B$  and  $\bar{A} = B$ . (p. 185 ∈ [1])
- $B$  is the **one-point compactification of**  $A \stackrel{def}{=} B$  is a *compactification* of  $A$  and  $B \setminus A$  is a *single point*. (p. 185 ∈ [1])
- The *one-point compactification* of  $\mathbb{R}$  is *homeomorphic* to a *circle*.  
(ex. 4 p. 185 ∈ [1])
- The *one-point compactification* of  $\mathbb{R}^2$  is *homeomorphic* to the *sphere*  $S^2$ .  
(ex. 4 p. 185 ∈ [1])
- **The Riemann Sphere** a.k.a. the **extended plane**  $\stackrel{def}{=} \mathbb{C} \cup \{\infty\}$ , when  $\mathbb{R}^2$  is seen as  $\mathbb{C}$ . (ex. 4 p. 185 ∈ [1])
- For *Hausdorff space*  $A$ :  $A$  is *locally compact*  $\Leftrightarrow \forall x \in A : \forall U$  *neighborhood* of  $x : \exists V$  *neighborhood* of  $x : \bar{V}$  is *compact* and  $\bar{V} \subset U$ . (thm. 29.2 p. 185 ∈ [1])
- For  $A$  *locally compact Hausdorff*,  $B \underset{\text{subspace}}{\subset} A$ :  
 $B \underset{\text{open}}{\subset} A$  or  $B \underset{\text{closed}}{\subset} A \Rightarrow B$  *locally compact*. (cor. 29.3 p. 185 ∈ [1])
- For *topological space*  $A$ :  
 $A$  is *homeomorphic* to an *open subspace* of a *compact Hausdorff space*  $\Leftrightarrow A$  *locally compact Hausdorff*. (cor. 29.4 p. 185 ∈ [1])

## 10 Nets

- Note: Nets do for *general topological spaces* what *sequences* do for *metrizable spaces*. (p. 187 ∈ [1])
- A **directed set**  $J \stackrel{def}{=} J$  is a set with partial order  $\preceq$  such that  $\forall \alpha, \beta \in J : \exists \gamma \in J : \alpha \preceq \gamma$  and  $\beta \preceq \gamma$ . (p. 187 ∈ [1])  
That  $\preceq$  is a *partial order* means that the following hold:

1.  $\forall \alpha : \alpha \preceq \alpha$
2.  $\alpha \preceq \beta \wedge \beta \preceq \alpha \Rightarrow \alpha = \beta$
3.  $\alpha \preceq \beta \wedge \beta \preceq \gamma \Rightarrow \alpha \preceq \gamma$

However all theorems here also hold if we only require the following of  $\preceq$ :

1.  $\forall \alpha : \alpha \preceq \alpha$
2.  $\alpha \preceq \beta \wedge \beta \preceq \gamma \Rightarrow \alpha \preceq \gamma$

Many mathematicians use the term *directed set* in this more general case.  
(exc. 29;12 p. 188 ∈ [1])

- Some *directed sets*: (exc. 29;1 p. 187 ∈ [1])
  1. Any *simply ordered set* with relation  $\leq$ .
  2. The *collection of all subsets* of a set  $S$ , *partially ordered by inclusion* (i.e.:  $A \preceq B$  when  $A \subset B$ ).
  3. The *collection  $\mathcal{A}$  of subsets* of a set  $S$  which is *closed under finite intersections, partially ordered by reverse inclusion* (i.e.:  $A \preceq B$  when  $A \supset B$ ).
  4. The *collection of all closed subsets* of a *topological space  $A$* , *partially ordered by inclusion*.
- For *directed set*  $J$  and  $K \subset J$ :  $K$  is **confinal in  $J$**   $\stackrel{def}{=}$  for each  $\alpha \in J$ , there exists  $\beta \in K$  such that  $\alpha \preceq \beta$ . (exc. 29;2 p. 187 ∈ [1])
- For *directed set*  $J$  and  $K \subset J$ :  $K$  is *confinal in  $J$*   $\Rightarrow$   $K$  *directed set*. (exc. 29;2 p. 187 ∈ [1])
- For *topological space  $A$* : A **net in  $A$**   $\stackrel{def}{=}$  a *function  $f : J \rightarrow A$*  from a *directed set  $J$*  into  $A$ .  
If  $\alpha \in J$ , we usually *denote  $f(\alpha)$*  by  $a_\alpha$ .  
We *denote the net  $f$*  itself by  $(a_\alpha)_{\alpha \in J}$  or merely  $(a_\alpha)$  if the index is understood. (exc. 29;3 p. 187 ∈ [1])
- **The net  $(a_\alpha)$  converges to  $a \in A$**  (written  $a_\alpha \rightarrow a$ )  $\stackrel{def}{=}$  for *each neighborhood  $U$*  of  $a$ , there exists  $\alpha \in J$  such that  $\alpha \preceq \beta \Rightarrow a_\beta \in U$ . (exc. 29;3 p. 187 ∈ [1])
- For *topological spaces  $A, B$* : If  $(x_\alpha)_{\alpha \in J} \rightarrow x$  in  $A$  and  $(y_\alpha)_{\alpha \in J} \rightarrow y$  in  $B$ , then  $(x_\alpha \times y_\alpha) \rightarrow x \times y$  in  $A \times B$ . (exc. 29;4 p. 187 ∈ [1])
- For *Hausdorff space  $A$* : A *net in  $A$*  *converges to at most one point*. (exc. 29;5 p. 187 ∈ [1])

- For *topological space*  $A$ , (Error:  $S \in A$ ?)  $S \subset A$ :  $x \in \overline{S} \Leftrightarrow$   
there *exists a net of points* of  $S$  *converging to*  $x$ . (exc. 29;6 p. 187 ∈ [1])
- For *topological spaces*  $A, B$ ,  $f : A \rightarrow B$ :  $f$  *continuous*  $\Leftrightarrow$   
for *every convergent net*  $(x_\alpha)$  in  $A$  *converging to*, say  $x$ ,  
the net  $(f(x_\alpha))$  *converges to*  $f(x)$ . (exc. 29;7 p. 188 ∈ [1])
- For *net*  $f : J \rightarrow A$  in *topological space*  $A$  and  $f(\alpha) = x_\alpha$ :  
**A subnet**  $f \circ g$  **of**  $(x_\alpha)$   $\stackrel{def}{=}$   
the *composite function*  $f \circ g : K \rightarrow A$ , where  
 $K$  is a *directed set* and  $g : K \rightarrow J$  is a *function* such that:
  1.  $i \preceq j \Rightarrow g(i) \preceq g(j)$
  2.  $g(K)$  is *confinal* in  $J$

(exc. 29;8 p. 188 ∈ [1])
- If  $(x_\alpha)$  *converges to*  $x$ , then so does *any subnet*. (exc. 29;8 p. 188 ∈ [1])
- For a *net*  $(x_\alpha)_{\alpha \in J}$  in  $A$ :  
 $x$  is an **accumulation point of the net**  $(x_\alpha)$   $\stackrel{def}{=}$   
for *each neighborhood*  $U$  of  $x$ , the set of those  $\alpha$  for which  $x_\alpha \in U$  is  
*confinal* in  $J$ . (exc. 29;9 p. 188 ∈ [1])
- The *net*  $(x_\alpha)$  has  $x$  as an *accumulation point*  $\Leftrightarrow$   
*some subnet* of  $(x_\alpha)$  *converges to*  $x$ . (exc. 29;9 p. 188 ∈ [1])
- $A$  is *compact*  $\Leftrightarrow$  *every net* in  $A$  has a *convergent subnet*. (exc. 29;10 p. 188 ∈ [1])
- For *topological group*  $G$ ,  $A, B \stackrel{\subset}{\subset} G$ :  
If  $A \stackrel{\subset}{\subset} G$  *closed* and  $B$  *compact*, then  $A \cdot B$  is *closed* in  $G$ . (exc. 29;11 p. 188 ∈ [1])

## 11 Countability and Separation Axioms

### 11.1 Needed Definitions for Countability

Premises:  $A, B$  *topological spaces*.

- $A$  has a **countable basis** at  $x \in A$   $\stackrel{def}{=}$   
there *exists* a *countable collection*  $\mathcal{B}$  of *neighborhoods* of  $x$  such that *each neighborhood* of  $x$  contains at least one of the elements of  $\mathcal{B}$ . (p. 190 ∈ [1])
- $B \subset A$  is **dense** in  $A$   $\stackrel{def}{=} \overline{B} = A$ . (p. 191 ∈ [1])

### 11.2 Countability Axioms

Premises:  $A, B$  *topological spaces*.

- $A$  **Satisfies the first countability axiom**  
a.k.a.  $A$  **is first countable**  $\stackrel{def}{=}$   
 $\forall x \in A : A$  has a *countable basis* at  $x$ . (p. 190 ∈ [1])
- $A$  **Satisfies the second countability axiom**  
a.k.a.  $A$  **is second countable**  $\stackrel{def}{=}$   
 $A$  has a *countable basis* for its *topology*. (p. 190 ∈ [1])
- $A$  is a **Lindelöf space**  $\stackrel{def}{=}$   
*Every open covering* of  $A$  has a *countable subcollection covering*  $A$ .  
(p. 192 ∈ [1])
- $A$  is **isseparable** (which has nothing to do with a *separation* of  $A$ )  $\stackrel{def}{=}$   
There *exists* a *countable subset* of  $A$  which is *dense* in  $A$ .  
(p. 192 ∈ [1])

### 11.3 Theorems About Countability Axioms

Premises:  $A, B$  *topological spaces*.

- Every *metrizable space* satisfies the *first countability axiom*. (p. 190 ∈ [1])
- For  $B \subset A$ :  
If there *exist* a *sequence of points* in  $B$  *converging* to  $x$ , then  $x \in \overline{B}$ .  
The *converse holds* if  $A$  is *first-countable*. (thm. 30.1 a) p. 190 ∈ [1])
- For  $f : A \rightarrow B$ : If  $f$  is *continuous*, then for *every convergent sequence*  $x_n \rightarrow x$  in  $A$ , the sequence  $f(x_n)$  *converges* to  $f(x)$ .  
The *converse holds* if  $A$  is *first-countable*. (thm. 30.1 b) p. 190 ∈ [1])
- A *first countable*  $\Rightarrow$  A *second countable*. (p. 190 ∈ [1])
- A *metrizable space* is *not necessarily second countable*. (p. 190 ∈ [1])
- $\mathbb{R}^\omega$  and  $\mathbb{R}^n$  are *second countable*. (ex. 1 p. 190 ∈ [1])
- $\mathbb{R}^\omega$  with the *uniform topology* is *first countable* but *not second countable*. (p. 190 ∈ [1])
- A *subspace* of a *first countable space* is *first countable*. (thm. 30.2 p. 191 ∈ [1])
- A *subspace* of a *second countable space* is *second countable*. (thm. 30.2 p. 191 ∈ [1])
- A *countable product* of *first countable spaces* is *first countable*. (thm. 30.2 p. 191 ∈ [1])
- A *countable product* of *second countable spaces* is *second countable*. (thm. 30.2 p. 191 ∈ [1])
- $A$  is a *second countable topological space*  $\Rightarrow A$  is *Lindelöf*. (thm. 30.3 a) p. 191 ∈ [1])
- $A$  is a *second countable topological space*  $\Rightarrow A$  is *separable*. (thm. 30.3 b) p. 191 ∈ [1])
- For *metrizable space*  $A$ :  $A$  *Lindelöf*  $\Leftrightarrow A$  *second countable*  $\Leftrightarrow A$  *separable*. (p. 192 ∈ [1])
- $\mathbb{R}_l$  is *first countable*, *Lindelöf* and *separable* but *not second countable*.  
I.e.  $\mathbb{R}_l$  satisfies *all countability axioms except second countability*. (ex. 3 p. 192 ∈ [1])
- A *product space* of 2 *Lindelöf spaces* need *not* be *Lindelöf*. (ex. 4 p. 193 ∈ [1])
- A *subspace* of a *Lindelöf space* need *not* be *Lindelöf*. (ex. 5 p. 193 ∈ [1])
- **The Sorgenfrey plane**  $\mathbb{R}_l^2 \stackrel{def}{=} \mathbb{R}_l \times \mathbb{R}_l$ .  
 $\mathbb{R}_l^2$  is *not Lindelöf*. (ex. 5 p. 193 ∈ [1])

## 11.4 Separation Axioms

Premises:  $A$  *topological space*.

- $(T_2)$   $A$  is **Hausdorff**  $\stackrel{def}{=}$   
 $\forall x, y \in A, x \neq y$  : there exist disjoint open sets containing  $x$  and  $y$   
 respectively. (p. 195  $\in$  [1])
- For *topological space*  $A$  where *one-point sets* are closed in  $A$ :  
 $(T_3)$   $A$  is **regular**  $\stackrel{def}{=}$   
 for each pair consisting of a point  $x$  and a closed set  $B$  disjoint from  $x$ ,  
 there exist disjoint open sets containing  $x$  and  $B$ , respectively. (p. 195  $\in$  [1])
- For *topological space*  $A$  where *one-point sets* are closed in  $A$ :  
 $(T_4)$   $A$  is **normal**  $\stackrel{def}{=}$   
 for each pair  $B, C$  of disjoint closed sets in  $A$ ,  
 there exist disjoint open sets containing  $B$  and  $C$ , respectively. (p. 195  $\in$  [1])
- For *topological space*  $A$  where *one-point sets* are closed in  $A$ :  
 $A$  is *regular*  $\Leftrightarrow$   
 $\forall x \in A : \forall U$  neighborhood of  $x : \exists V$  neighborhood of  $x : \overline{V} \subset U$ .  
 (lemma 31.1a p. 196  $\in$  [1])
- For *topological space*  $A$  where *one-point sets* are closed in  $A$ :  
 $A$  is *normal*  $\Leftrightarrow$   
 $\forall B \stackrel{C}{\subset} A : \forall U \stackrel{C}{\subset} A$ , where  $B \subset U : \exists V \stackrel{C}{\subset} A$ , where  $B \subset V : \overline{V} \subset U$ .  
 (lemma 31.1b p. 196  $\in$  [1])
- $A$  *normal*  $\Rightarrow A$  *regular*. (p. 195  $\in$  [1])
- $A$  *regular*  $\Rightarrow A$  *Hausdorff*. (p. 195  $\in$  [1])
- A *subspace* of a *Hausdorff space* is *Hausdorff*. (thm. 31. 2a p. 196  $\in$  [1])
- A *product space* of *Hausdorff spaces* is *Hausdorff*. (thm. 31. 2a p. 196  $\in$  [1])
- A *subspace* of a *regular space* is *regular*. (thm. 31. 2a p. 196  $\in$  [1])
- A *product space* of *regular spaces* is *regular*. (thm. 31. 2a p. 196  $\in$  [1])
- Warning: *Subspaces* or *products* of *normal spaces* need *not* be *normal*.  
 (p. 196  $\in$  [1])
- $\mathbb{R}_K$  is *Hausdorff* but *not regular*. (ex. 1 p. 197  $\in$  [1])
- $\mathbb{R}_l$  is *normal*. (ex. 2 p. 198  $\in$  [1])
- The *Sorgenfrey plane*  $\mathbb{R}_l^2$  is *not normal*. (ex. 3 p. 198  $\in$  [1])

## 12 Normal Spaces

Premises:  $A$  *topological space*.

- Every *regular space* with a *countable basis* is *normal*. (thm. 32.1 p. 200 ∈ [1])
- Every *metrizable space* is *normal*. (thm. 32.2 p. 202 ∈ [1])
- Every *compact Hausdorff space* is *normal*. (thm. 32.3 p. 202 ∈ [1])
- Every *order topology* is *normal*. (thm. 32.4, ex. 39 p. 202 ∈ [1])
- If  $J$  is *uncountable*, the *product space*  $\mathbb{R}^J$  is *not normal*. (ex. 1 p. 203 ∈ [1])
- $S_\Omega \times \overline{S_\Omega}$  is *not normal* (ex. 2 p. 203 ∈ [1])  
but is *completely regular*. (ex. 1 p. 212 ∈ [1])
- $(T_5)$   $A$  is **completely normal**  $\stackrel{def}{=}$   
*every subspace of  $A$  is normal*. (exc. 32;6 p. 205 ∈ [1])
- *The Urysohn Lemma:*  
For *normal topological space*  $A$  and  $B, C$  *disjoint closed subsets of  $A$ :*  
If  $[a, b]$  is a *closed interval of the real line*, then  
*there exists a continuous map*  $f : A \rightarrow [a, b]$  such that  
 $(\forall x \in B : f(x) = b)$  and  $(\forall x \in C : f(x) = a)$ . (thm. 33.1 p. 207 ∈ [1])
- The *converse of the Urysohn Lemma* is *trivial*. (p. 211 ∈ [1])
- For  $B, C \subset A$ :  
 $A$  and  $B$  can be **separated by a continuous function**  $\stackrel{def}{=}$   
*there exist a continuous function*  $f : A \rightarrow [0, 1]$  such that  
 $f(A) = \{0\}$  and  $f(B) = \{1\}$ . (p. 211 ∈ [1])
- $(T_{3\frac{1}{2}})$   $A$  is **completely regular**  $\stackrel{def}{=}$   
*one-point sets are closed in  $A$  and*  
*for each point  $x_0$  and each closed set  $C$  where  $x_0 \notin C$ , then*  
*there exists a continuous function*  $f : A \rightarrow [0, 1]$  such that  
 $f(x_0) = 1$  and  $f(C) = \{0\}$ . (p. 211 ∈ [1])
- $A$  *normal*  $\Rightarrow$   $A$  *completely regular*. (p. 211 ∈ [1])
- $A$  *completely regular*  $\Rightarrow$   $A$  *regular*. (p. 211 ∈ [1])
- *Subspaces of completely regular spaces are completely regular.*  
(thm. 33.2 p. 211 ∈ [1])
- *Products of completely regular spaces are completely regular.*  
(thm. 33.2 p. 211 ∈ [1])
- $A$  is **perfectly normal**  $\stackrel{def}{=}$   
 $A$  is *normal* and *every closed set in  $A$  is a  $G_\delta$ -set in  $A$* . (exc. 33;6 p. 213 ∈ [1])
- *Urysohn metrization theorem:*  
Every *regular space*  $A$  with a *countable basis* is *metrizable*.  
(thm. 34.1 p. 215 ∈ [1])

- **Imbedding Theorem:**

For *topological space*  $A$  where *one-point sets* are *closed*:

If  $\{f_\alpha\}_{\alpha \in J}$  is an *indexed family of continuous functions*  $f_\alpha : A \rightarrow \mathbb{R}$  *satisfying* that for *each point*  $x_0 \in A$  and *each neighborhood*  $U$  of  $x_0$ , there is an *index*  $\alpha$  such that  $f_\alpha$  is *positive* at  $x_0$  and *vanishes outside*  $U$ . Then the *function*  $F : A \rightarrow \mathbb{R}^J$  defined by  $F(x) = (f_\alpha(x))_{\alpha \in J}$  is an *embedding* of  $A$  into  $\mathbb{R}^J$ .

If  $f_\alpha$  maps into  $[0, 1]$  for each  $\alpha$ , then  $F$  *imbeds*  $A$  in  $[0, 1]^J$ .

(thm. 34.2 p. 217  $\in$  [1])

$\{f_\alpha\}_{\alpha \in J}$  is said to *separate points from closed sets* in  $A$ . (p. 218  $\in$  [1])

- $A$  is *completely regular*  $\Leftrightarrow$

$A$  is *homeomorphic* to a *subspace* of  $[0, 1]^J$  for *some*  $J$ .

(thm. 34.3 p. 218  $\in$  [1])

## 12.1 Tietze Extension Theorem

Premises:  $A$  *normal space*,  $B$  *closed subspace*  $A$ .

- Any *continuous map*  $f : B \rightarrow [a, b]$  may be *extended* to a *continuous map*  $f' : A \rightarrow [a, b]$ , where  $[a, b]$  is a *closed interval* of  $\mathbb{R}$ .

(thm. 35.1a p. 219  $\in$  [1])

- *Tietze extension theorem:* Any *continuous map*  $f : B \rightarrow \mathbb{R}$  may be *extended* to a *continuous map*  $f' : A \rightarrow \mathbb{R}$ . (thm. 35.1b p. 219  $\in$  [1])

## 13 Imbeddings of Manifolds

Premises:  $A$  *topological space*.

- **An  $m$ -manifold**  $A \stackrel{def}{=} \overline{\quad}$  a Hausdorff space  $A$  with a countable basis such that each point  $x \in A$  has a neighborhood that is homeomorphic with an open subset of  $\mathbb{R}^m$ .  
(p. 225 ∈ [1])  
 Note: A 1-manifold is often called a *curve* and a 2-manifold is often called a *surface*. (p. 225 ∈ [1])
- **The support of a function**  $\phi : A \rightarrow \mathbb{R} \stackrel{def}{=} \overline{\phi^{-1}(\mathbb{R} \setminus \{0\})}$ . Thus if  $x \notin$  support of  $\phi$ , there is some neighborhood of  $x$  where  $\phi$  vanishes. (p. 225 ∈ [1])
- For finite indexed open covering  $\{U_1, \dots, U_n\}$  of  $A$ :  
 An indexed family of continuous functions  $\phi_i : A \rightarrow [0, 1]$  where  $i \in \{1, \dots, n\}$  is a **partition of unity dominated by  $\{U_i\}$**   $\stackrel{def}{=} \overline{\quad}$ 
  1.  $\forall i \in \{1, \dots, n\} : (\text{support } \phi_i) \subset U_i$
  2.  $\forall x \in A : \sum_{i=1}^n \phi_i(x) = 1$(p. 225 ∈ [1])
- **Existence of finite partitions of unity** for normal topological space  $A$ :  
 If  $\{U_1, \dots, U_n\}$  is a finite open covering of  $A$ , then there exists a partition of unity dominated by  $\{U_i\}$ . (thm. 36.1 p. 225 ∈ [1])
- If  $A$  is a compact  $m$ -manifold, then  $A$  can be imbedded in  $\mathbb{R}^N$  for some positive integer  $N$ . (thm. 36.2 p. 226 ∈ [1])
- For set  $S$ : If  $\mathcal{A}$  is a collection of subsets of  $S$  with the finite intersection property, then there exists a collection  $\mathcal{D}$  of subsets of  $S$  such that  $\mathcal{D}$  contains  $\mathcal{A}$ , and  $\mathcal{D}$  has the finite intersection property and no other collection of subsets of  $S$  that properly contains  $\mathcal{D}$  has this property.  
 I.e.:  $\mathcal{D}$  is maximal w.r.t. the finite intersection property.  
(lemma 37.1 p. 233 ∈ [1])
- For a set  $S$  and a collection  $\mathcal{D}$  of subsets of  $S$  that is maximal w.r.t. the finite intersection property:
  1. Any finite intersection of elements of  $\mathcal{D}$  is an element of  $\mathcal{D}$
  2. If  $C$  is a subset of  $S$  that intersects every element of  $\mathcal{D}$ , then  $C$  is an element of  $\mathcal{D}$ .(lemma 37.2 p. 234 ∈ [1])

## 14 Stone-Čech Compactification

Premises:  $A$  *topological space*.

- **A compactification  $B$  of  $A$**   $\stackrel{def}{=}$   
a *compact Hausdorff space*  $B$  having  $A$  as a *subspace* such that  $\overline{A} = B$ .  
(p. 237 ∈ [1])
- **Two compactifications  $B_1$  and  $B_2$  of  $A$  are equivalent**  $\stackrel{def}{=}$   
there *exists* a *homeomorphism*  $h : B_1 \rightarrow B_2$  such that  $\forall x \in A : h(x) = x$ .  
(p. 237 ∈ [1])
- $A$  has a compactification  $B \Leftrightarrow A$  is *completely regular*. (p. 237 ∈ [1])
- For *Hausdorff space*  $C$ : If  $h : A \rightarrow C$  is an *embedding* of  $A$  in  $C$ , then there *exists* a *corresponding compactification*  $B$  of  $A$ .  
 $B$  has the *property* that there *exists* an *embedding*  $H : B \rightarrow C$  that *equals*  $h$  on  $A$ .  
The *compactification*  $B$  is *uniquely determined* up to *equivalence*.  
 $B$  is called the **compactification induced by the imbedding  $h$** .  
(lemma 38.1 p. 237 ∈ [1])
- General note: There are *many compactifications*.  
*Stone-Čech* is in some sense the *maximal compactification* while the *one-point compactification* is in some sense the *minimal compactification*.  
(p. 237-238 ∈ [1])
- Compactification examples:
  - $[0, 1]$  is a *compactification* of  $]0, 1[$ ,  
obtained by *adding one point* to *each end* of  $]0, 1[$ . (ex. 2 p. 238 ∈ [1])
  - The *unit circle*  $S^1$  is *equivalent* to  
the *one-point compactification* of  $]0, 1[$ . (ex. 1 p. 238 ∈ [1])
  - Let  $h : ]0, 1[ \rightarrow [-1, 1]^2$  be the map  $h(x) = (x, \sin(1/x))$ .  
The space  $Y_0 = h(]0, 1[)$  is the *Topologist's Sine Curve*.  
(see (ex. 7 paragraph 24 ∈ [1])).  
The *imbedding*  $h$  gives a *compactification* of  $]0, 1[$   
by *adding a point* and a *line segment* to  $]0, 1[$ . (ex. 3 p. 238 ∈ [1])
- **Compactification with extension condition:**  
For *completely regular space*  $A$ :  
There *exists* a *compactification*  $B$  of  $A$  having the *property* that *every bounded continuous map*  $f : A \rightarrow \mathbb{R}$  *extends uniquely* to a *continuous map* of  $B$  into  $\mathbb{R}$ . (thm. 38.2 p. 239 ∈ [1])
- For *completely regular space*  $A$ ,  $S \subset A$ ,  
*Hausdorff space*  $C$ ,  $f : S \rightarrow C$  *continuous*:  
There is *at most one* extension of  $f$  to a *continuous function*  $g : \overline{S} \rightarrow C$ .  
(lemma 38.3 p. 240 ∈ [1])
- For *completely regular space*  $A$ , *compactification*  $B$  of  $A$  satisfying the *extension condition*, *compact Hausdorff space*  $C$ :  
Given *any continuous map*  $f : A \rightarrow C$ , the map  $f$  *extends uniquely* to a *continuous map*  $g : B \rightarrow C$ . (thm. 38.4 p. 240 ∈ [1])

- For *completely regular space*  $A$ :  
If  $B_1$  and  $B_2$  are *two compactifications* of  $A$  satisfying the *extension theorem*, then  $B_1$  and  $B_2$  are *equivalent*. (thm. 38.5 p. 240 ∈ [1])
- For *completely regular space*  $A$ :  
**The Stone-Čech compactification  $\beta(A)$  of  $A$**   $\stackrel{def}{=}$   
the *compactification* of  $A$  satisfying the *extension condition*.  
 $\beta(A)$  is *characterized* by the fact that *any continuous map*  $f : A \rightarrow C$   
into a *compact Hausdorff space*  $C$  *extends uniquely* to a *continuous map*  
 $g : \beta(A) \rightarrow C$ . (p. 241 ∈ [1])

## 15 Questions

- Why does  $]a, +\infty[ \cup B$  and  $] - \infty, a[ \cup B$  form a *subbasis* for the *subspace topology* on  $B$ ? (proof of thm. 16.4)

## References

- [1] James R. Munkres. *Topology, second edition*, Pearson Education (Prentice Hall?) 2000.