Real Numbers Summary

Ánoq of the Sun, Hardcore Processing *

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1 The Real Numbers

- $\mathbb{R}$ is an ordered *body (?)* $\underset{def}{=}$ it has the operations $+, \cdot, \leq$ with a lot of well-known axioms (p. 1 [1])

1.1 Limited Subsets of $\mathbb{R}$

- $A \subseteq \mathbb{R}$ is *upwards limited* $\underset{def}{=}$ $\exists s \in \mathbb{R} : \forall x \in A : x \leq s$. s is called an *upper limit* or a *majorant (?)* for A (def. 1.1 [1])
- $A \subseteq \mathbb{R}$ is *downwards limited* $\underset{def}{=}$ $\exists m \in \mathbb{R} : \forall x \in A : x \geq m$. m is called a *lower limit* or a *minorant (?)* for A (def. 1.1 [1], p. 6.3 [2])
- $A \subseteq \mathbb{R}$ is *limited* $\underset{def}{=}$ A upwards and downwards limited (def. 1.1 [1])

This definition of *limited* is equivalent to the one for metric spaces (p. 1 [1])

2 Supremum / Infimum Properties

- The supremum / infimum properties are also called *the completeness axiom* (p. 6.3 [2])
- Any non-empty upwards limited set $A \subseteq \mathbb{R}$ has a smallest upper limit $\sup A$ (the completeness axiom). From this follows:

  1. $\forall x \in A : x \leq \sup A$
  2. $s \in \mathbb{R}$ where $\forall x \in A : x \leq s \Rightarrow \sup A \leq s$

  Any downwards limited set $\emptyset \neq B \subseteq \mathbb{R}$ has largest lower limit $\inf B$

  1. $\forall x \in B : x \geq \inf B$
  2. $m \in \mathbb{R}$ where $\forall x \in B : x \geq m \Rightarrow \inf B \geq m$

  And we define:

  1. A not upwards limited $\Rightarrow \sup A \underset{def}{=} \infty$
  2. $\sup \emptyset \underset{def}{=} -\infty$

3. B not downwards limited $\Rightarrow \inf B \underset{def}{=} -\infty$
4. $\inf \emptyset \underset{def}{=} \infty$

(ax. 1.2 [1], p. 6.3 [2])

2.1 Some Theorems on Supremum / Infimum

- $A \subseteq \mathbb{R}$ does not always contain $\sup A$ and $\inf A$, but:

  1. $\emptyset \neq A \subseteq \mathbb{R}$ upwards ltd. $\Rightarrow \forall \epsilon > 0 : \exists x \in A : \sup A - \epsilon < x \leq \sup A$
  2. $\emptyset \neq B \subseteq \mathbb{R}$ downwards ltd. $\Rightarrow \forall \epsilon > 0 : \exists x \in B : \inf B + \epsilon > x \geq \inf B$

  (rem. 1.4 [1])

- $\forall x \in \mathbb{R} : \exists n \in \mathbb{Z} : x \leq n$ (principle of Archimedes) (thm. 1.6 [1])
- $\forall a, b \in \mathbb{R} : \exists r \in \mathbb{Q} : r \in \,]a, b[$ (cor. 1.7 [1])

2.2 Theorems on Sequences and Sets Using sup / inf

- A sequence $(x_n)_{n=1}^{\infty}$ is *growing* $x_1 \leq x_2 \leq x_3 \leq \cdots$ (p. 2 [1])
- A sequence $(x_n)_{n=1}^{\infty}$ is *falling* $x_1 \geq x_2 \geq x_3 \geq \cdots$ (p. 2 [1])
- The *set of a sequence* $(x_n)_{n=1}^{\infty} \underset{def}{=}$ the set $\{x_1, x_2, \dots\}$ (p. 2 [1])
- $(x_n)_{n=1}^{\infty}$ *upwards limited* $\underset{def}{=}$ the set of (x_n) upwards ltd. (p. 2 [1])
- $(x_n)_{n=1}^{\infty}$ *downwards limited* $\underset{def}{=}$ the set of (x_n) downwards ltd. (p. 2 [1])
- Any *upwards limited growing* sequence (x_n) of real numbers *converges* (thm 1.5 [1])
- Any *downwards limited falling* sequence (x_n) of real numbers *converges* (thm 1.5 [1])

3 Limes Superior and Limes Inferior

- *Limes superior* for any real sequence $(x_n)_{n=1}^{\infty}$: $\limsup\limits_{n\to\infty} x_n \underset{def}{=} \inf\limits_{n\in\mathbb{N}} s_n = \inf\limits_{n\in\mathbb{N}} \sup\limits_{k\geq n} x_k$, where $s_n = \sup\limits_{k\geq n} x_k$ (p. 5 [1])
- $s_n = \sup\limits_{k\geq n} x_k$ is a *falling* sequence since we take sup. of gradually smaller sets (p. 5 [1])
- $s_n = \sup\limits_{k\geq n} x_k$ downwards limited $\Rightarrow (s_n)$ converges and then: $\limsup\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \sup\limits_{k\geq n} x_k$ (p. 5 [1])
- *Limes inferior* for any real sequence $(x_n)_{n=1}^{\infty}$: $\liminf\limits_{n\to\infty} x_n \underset{def}{=} \sup\limits_{n\in\mathbb{N}} t_n = \sup\limits_{n\in\mathbb{N}} \inf\limits_{k\geq n} x_k$, where $t_n = \inf\limits_{k\geq n} x_k$ (p. 5 [1])
- $t_n = \inf\limits_{k\geq n} x_k$ is a *growing* sequence since we take inf. of gradually smaller sets (p. 5 [1])

- $t_n = \inf_{k \geq n} x_k$ upwards limited $\Rightarrow (t_n)$ converges and then: $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} t_n = \lim_{n\to\infty} \inf_{k \geq n} x_k$ (p. 5 [1])

- For any real sequence $(x_n)_{n=1}^{\infty}$ where $s_n = \sup_{k \geq n} x_k$ and $t_n = \inf_{k \geq n} x_k$: $t_1 \leq t_2 \leq \cdots \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \cdots \leq s_2 \leq s_1$ (p. 5 [1])

- A real sequence $(x_n)_{n=1}^{\infty}$ converges towards $a \in \mathbb{R} \Leftrightarrow$ $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = a$ (thm. 3.2 [1])

- $(x_n)_{n=1}^{\infty}$ limited real sequence $\Rightarrow \limsup x_n$ and $\liminf x_n$ are both finite real numbers. More specificly: $a \leq x_n \leq b$ for all $n \in \mathbb{N} \Rightarrow a \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} \leq b$ (lemma 3.3 [1])

- For any real sequence $(x_n)_{n=1}^{\infty}$:

  1. $a = \limsup x_n \in \mathbb{R}$ finite number $\Rightarrow$
     $\forall \epsilon > 0 : \{n \in \mathbb{N} \mid |x_n - a| < \epsilon\}$ infinite and $\{n \in \mathbb{N} \mid x_n \geq a + \epsilon\}$ finite
  2. $a = \limsup x_n \in \mathbb{R}$ finite number $\Rightarrow$
     a is the largest condensation point for (x_n)
  3. $b = \liminf x_n \in \mathbb{R}$ finite number $\Rightarrow$
     $\forall \epsilon > 0 : \{n \in \mathbb{N} \mid |x_n - b| < \epsilon\}$ infinite and $\{n \in \mathbb{N} \mid x_n \geq b - \epsilon\}$ finite
  4. $b = \liminf x_n \in \mathbb{R}$ finite number $\Rightarrow$
     a is the smallest condensation point for (x_n)

  (thm. 3.4 [1])

- For any real sequence $(x_n)_{n=1}^{\infty}$:

  1. $\limsup x_n \in \mathbb{R}$ finite number $\Rightarrow$
     there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ where $\lim_{n\to\infty} x_{n_k} = \limsup x_n$
  2. $\liminf x_n \in \mathbb{R}$ finite number $\Rightarrow$
     there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ where $\lim_{n\to\infty} x_{n_k} = \liminf x_n$

  (cor. 3.5 [1])

- Any real sequence in a limited interval has a condensation point (Bolzano-Weierstrass) (thm. 3.6 [1])

- Any Cauchy sequence in $\mathbb{R}$ converges (Ordinary Principle of Convergence) (thm. 3.7 [1])

References

[1] Mikael Rørdam (?). *Om de Reelle Tal - Supplerende Noter til 2AN*, Maths Department University of Copenhagen 2001.

[2] Christian Berg. *Metriske Rum*, Matematisk Afdeling Københavns Universitet 1997.