

Probability Summary

Ánoq of the Sun, Hardcore Processing *

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1 Probability

1.1 Intro, Notation, Terminology

Premises: Ω *sample space*, \mathcal{A} *all possibilities of Ω*

- **The *sample space* Ω** (greek: δ, χ .) $\stackrel{def}{=}_{\langle p. 21 \in [1] \rangle}$
The *set of all the possible results* of an *experiment of chance* (greek: π, τ .)
 Ω may be *discrete* (i.e. *finite* or *countable*) or *continuous*, i.e.:
 $\Omega = \{\omega_1, \dots, \omega_n\}$: *finite (discrete)*
 $\Omega = \{\omega_1, \omega_2, \dots\}$: *countably infinite (discrete)*
 $\Omega = \{\omega \in \mathbb{R} \mid 0 \leq \omega \leq 1\}$: *Over-countably infinite (continuous)*
- **For $\omega \in \Omega$, ω is called a *sample point* / *basic possibility*** $\langle p. 21 \in [1] \rangle$
- **The *set of all possibilities* \mathcal{A} of Ω** $\stackrel{def}{=}_{\langle def. 2.1 p. 22 \in [1] \rangle}$
Either the *power set* of Ω : $\mathcal{A} = \mathcal{P}(\Omega)$ (always the case for *discrete* Ω)
or a *subset* of the *power set*: $\mathcal{A} \subset \mathcal{P}(\Omega)$ (only for *continuous* Ω)
In general \mathcal{A} *must also* be a σ -*field* or a σ -*algebra*, which means:
 1. $\Omega \in \mathcal{A}$
 2. $(A \in \mathcal{A}) \Rightarrow (\complement A \in \mathcal{A})$
 3. $(\forall \nu \in J : A_\nu \in \mathcal{A}) \Rightarrow (\bigcap_{\nu \in J} A_\nu \in \mathcal{A})$ for *countable index set* JFrom these it follows that e.g. also *countable unions* and \emptyset are in \mathcal{A} .
- **For $A \subset \Omega$, A is called a *possible outcome*** $\langle def. 2.1 p. 22 \in [1] \rangle$
E.g.: For a *die roll* of a *6-sided die*: $\Omega = \{1, \dots, 6\}$, $\omega = 4$, $A = \{1, 2, 3\}$
- **For $A \in \mathcal{A}$, A is a *possible outcome*** $\langle def. 2.1 p. 22 \in [1] \rangle$

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1.2 Different Probability Theories

- **Theory of Laplace** (for discrete Ω):
If all *possible outcomes* have the same probability.
Assume that a *possible outcome* A results from $N(A)$ elements of Ω , then $P(A) = \frac{N(A)}{N}$, when $N = N(\Omega)$. (p. 26 ∈ [1])
- **Geometric Probabilities**: Let Ω be an *over-countable set* with *measure* $\mu(\Omega)$ (*area, volume, etc.*) and let $A \subset \Omega$ with $\mu(A)$ being the measure.
If all *possibilities* of Ω have the *same probabilities*, then $P(A) = \frac{\mu(A)}{\mu(\Omega)}$ (p. 27 ∈ [1])
- **Theory of Von Mises**:
If the *relative frequency* of the *possible outcome* A is $R_\nu(A) = \frac{m_\nu(A)}{\nu}$, then $P(A) = \lim_{\nu \rightarrow \infty} R_\nu(A)$. (p. 40 ∈ [1])
- For both Laplace and Von Mises Theory the *axioms of Kolmogorov* holds with *finite additivity* instead of σ -*additivity*. (p. 26 ∈ [1])
- **Homeomorphic probabilities** $\stackrel{def}{=} \forall j \in \{1, \dots, N\} : P(\{\omega_j\}) = \frac{1}{N}$ (rem 6.1 (i) p. 42 ∈ [1])

1.3 Counting the Number of Elements $N(A)$ of a Possible Outcome A

- For a *finite sample space* Ω with ν *elements*: The *power set* $\mathcal{P}(\Omega)$ is:
$$\binom{\nu}{0} + \binom{\nu}{1} + \dots + \binom{\nu}{\nu} = (1 + 1)^\nu = 2^\nu$$
- For *finite disjoint* A_1, A_2, \dots, A_n :
$$N(A_1 \cup A_2 \cup \dots \cup A_n) = N(A_1) + N(A_2) + \dots + N(A_n)$$

(*summation principle* of combinatorics) (p. 29 ∈ [1])
- For *finite disjoint* A_1, A_2, \dots, A_n :
$$N(A_1 \times A_2 \times \dots \times A_n) = N(A_1)N(A_2) \dots N(A_n)$$

(*multiplication principle* of combinatorics) (p. 29 ∈ [1])

1.4 The Axiomatic Foundation by Kolmogorov

Let Ω be the *sample space* of an *experiment of randomness* with \mathcal{A} being the σ -*algebra of possibilities*.

Probability is a *set function* $P : \mathcal{A} \rightarrow \mathbb{R}$ which *complies with the following*:

- Axioms: (def. 6.1 p. 41 ∈ [1])
 1. $\forall A \in \mathcal{A} : P(A) \geq 0$
 2. $P(\Omega) = 1$
 3. $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for *pairwise disjoint* $A_i \in \mathcal{A}$. (σ -*additivity*)

The probability space $\stackrel{def}{=} (\Omega, \mathcal{A}, P)$

- Theorems (from the axioms), where $A, B \in \mathcal{A}$:
 1. $P(\emptyset) = 0$ (thm. 6.1 (i) p. 43 ∈ [1])
 2. $\sum_{i=1}^{\nu} P(A_i) = P(\cup_{i=1}^{\nu} A_i)$ for *pairwise disjoint* $A_i \in \mathcal{A}$
(*finite additivity*) (thm. 6.1 (ii) p. 43 ∈ [1])
 3. $P(\complement A) = 1 - P(A)$ (thm. 6.1 (iii) p. 43 ∈ [1])
 4. $P(A \setminus B) = P(A) - P(A \cap B)$
And if $B \subset A$, then $P(A) - P(B)$ (thm. 6.1 (iv) p. 43 ∈ [1])
 5. $0 \leq P(A) \leq 1$ (thm. 6.2 p. 45 ∈ [1])
 6. For $A \subset B : P(A) \leq P(B)$ - i.e. P is a *growing set function*
(thm. 6.2 p. 45 ∈ [1])
 7. For $A_i \in \mathcal{A} : P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$ (thm. 6.3 p. 48 ∈ [1])
(*Boole's inequality* / σ -*subadditivity* / *countable subadditivity*)
- Theorems similar to the *inclusion principle* (and it's inversion), $A, B \in \mathcal{A}$:
 1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (thm. 7.1p. 53 ∈ [1])
 2. $P(\complement A \cap \complement B) = 1 - P(A) - P(B) + P(A \cap B)$ (cor. 7.1p. 53 ∈ [1])
 3. Poincarè's Theorem: For $A_1, \dots, A_{\nu} \in \mathcal{A}$: (thm. 7.2 p. 54 ∈ [1])
 $P(A_1 \cup \dots \cup A_{\nu}) = S_{\nu,1} - S_{\nu,2} + \dots + (-1)^{\nu-1} S_{\nu,\nu}$, where
 $S_{\nu,1} = \sum_{i=1}^{\nu} P(A_i)$,
 $S_{\nu,2} = \sum_{i=1}^{\nu-1} \sum_{j=i+1}^{\nu} P(A_i \cap A_j)$,
 \vdots
 $S_{\nu,r} = \sum P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_r})$, where the *sum* comes from
all the coices of the r *indices* $\{i_1, \dots, i_r\}$ *out of* $\{1, \dots, \nu\}$,
 \vdots
 $S_{\nu,\nu} = P(A_1 \cap A_2 \cap \dots \cap A_{\nu})$
 4. For $A_1, \dots, A_{\nu} \in \mathcal{A}$: (cor. 7.2 p. 54 ∈ [1])
 $P(\complement A_1 \cap \dots \cap \complement A_{\nu}) = 1 - S_{\nu,1} + S_{\nu,2} - \dots + (-1)^{\nu} S_{\nu,\nu}$, where
 $S_{q,r}$ are as in *Poincarè's theorem*.

1.5 Special Cases

- For $A_1, \dots, A_\nu \in \mathcal{A}$: **The A_i are *exchangeable*** $\stackrel{def}{=} \langle \text{def. 7.1 p.57} \in [1] \rangle$
 $\forall r \in \{1, \dots, \nu\} : P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_r}) = P_r$ for
any choice of indices $\{i_1, \dots, i_r\}$.
 I.e.: $P(A_1) = P(A_2) = \dots = P(A_\nu)$,
 $P(A_1 \cap A_2) = P(A_1 \cap A_3) = \dots = P(A_{\nu-1} \cap A_\nu)$,
 $P(A_1 \cap A_2 \cap \dots \cap A_r) = P(A_2 \cap A_3 \cap \dots \cap A_{r+1}) = \dots$
- For *exchangeable* $A_1, \dots, A_\nu \in \mathcal{A}$: $\langle \text{cor. 7.3 p.57} \in [1] \rangle$

1. $P(A_1 \cup \dots \cup A_\nu) = \binom{\nu}{1} P(A_1) - \binom{\nu}{2} P(A_1 \cap A_2) + \dots +$
 $(-1)^{r-1} \binom{\nu}{r} P(A_1 \cap \dots \cap A_r) + \dots + (-1)^{\nu-1} \binom{\nu}{\nu} P(A_1 \cap \dots \cap A_\nu)$
2. $P(\complement A_1 \cap \dots \cap \complement A_\nu) =$
 $1 - \binom{\nu}{1} P(A_1) + \dots + (-1)^\nu \binom{\nu}{\nu} P(A_1 \cap \dots \cap A_\nu)$

1.6 Bound Probabilities (A Posteriori Probabilities)

Premises: (Ω, \mathcal{A}, P) **probability space**, $A, B \in \mathcal{A}$, $P(A) > 0$ c

- **The probability of B bound by A** (when $P(A) > 0$) $\stackrel{def}{=} \langle \text{def. 1.1 p. 80} \in [1] \rangle$
 the *set function* $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{N(A \cap B)}{N(A)}$ $\langle \text{def. 1.1 p. 80} \in [1] \rangle$
- The *axioms of Kolmogorov* also hold for $P(B|A)$: $\langle \text{def. 1.1 p. 80} \in [1] \rangle$
 1. $P(B|A) \geq 0$
 2. $P(\Omega|A) = 1$
 3. $P(\cup_{i=1}^{\infty} B_i|A) = \sum_{i=1}^{\infty} P(B_i|A)$ for *pairwise disjoint* $B_i \in \mathcal{A}$

Thus $P(\cdot|A)$ is a *probability function*, so the usual theorems apply. e.g:

$\langle \text{rem. 1.1. p. 80} \in [1] \rangle$

1. $P(\complement B|A) = 1 - P(B|A)$
 Warning: $P(B|\complement A) = 1 - P(B|A)$ does *not hold!*
2. $P(B_1 \cup B_2|A) = \frac{P(B_1 \cap A) + P(B_2 \cap A) - P(B_1 \cap B_2 \cap A)}{P(A)}$

- **Multiplication theorem:** For $\forall i \in \{1, \dots, \nu\} : A_i \in \mathcal{A}$ and
 $P(A_1 \cap A_2 \cap \dots \cap A_{\nu-1}) > 0$: $\langle \text{thm. 1.1 p. 81} \in [1] \rangle$
 $P(A_1 \cap A_2 \cap \dots \cap A_\nu) =$
 $P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \dots P(A_\nu|A_1 \cap \dots \cap A_{\nu-1})$
- **Theorem of all probabilities:** For *partition* $\{A_i \mid i \in \mathbb{N}\}$ of *sample space* Ω
 where $\forall i \in \mathbb{N} : P(A_i) > 0$ we have:
 $\forall B \in \mathcal{A} : P(B) = \sum_{i=1}^{\infty} P(A_i)P(B|A_i)$. $\langle \text{thm. 1.2 p. 84} \in [1] \rangle$
- **Bayes theorem:** For *partition* $\{A_i \mid i \in \mathbb{N}\}$ of *sample space* Ω where
 $\forall i \in \mathbb{N} : P(A_i) > 0$ and a *possible outcome* $B \in \mathcal{A}$ where $P(B) > 0$:
 $\forall i \in \mathbb{N} : P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{k=1}^{\infty} P(A_k)P(B|A_k)}$. $\langle \text{thm. 1.3 p. 84} \in [1] \rangle$
- **The theorem of all probabilities and Bayes theorem** also hold for
finite partitions. $\langle \text{rem. 1.3 p. 85} \in [1] \rangle$

1.7 Stochastic Independence

Premises: (Ω, \mathcal{A}, P) *probability space*, $A, B \in \mathcal{A}$

- For A, B *disjoint*: $P(B|A) = 0$ (p. 91 (i) ∈ [1])
- For $A \subset B$ (so: $A \cap B = B$): $P(B|A) = 1$ (p. 91 (ii) ∈ [1])
- For $B \subsetneq A \subsetneq \Omega$ (so: $A \cap B = B$): $P(B|A) > P(B)$ (p. 91 (iii) ∈ [1])
- For $\complement B \subsetneq A \subsetneq \Omega$ (so: $A \cap \complement B = \complement B$): $P(B|A) < P(B)$ (p. 91 (iv) ∈ [1])
- For $P(A), P(B) > 0$: $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$ (p. 91 ∈ [1])
- B is *stochastically independent* of $A \stackrel{def}{=} P(B|A) = P(B)$ (p. 91 (v) ∈ [1])
- If B is *stochastically independent* of A (or vice versa) then:
 $P(A \cap B) = P(A)P(B)$ (p. 91 ∈ [1])
- A and B are *stochastically independent* $\stackrel{def}{=} P(A \cap B) = P(A)P(B)$ (def. 2.1 p. 92 ∈ [1])
- If A and B are *stochastically independent*, then so is:
 A and $\complement B$, $\complement A$ and B , $\complement A$ and $\complement B$. (p. 93 ∈ [1])

2 Combinatorics

- **Summation Principle:** Assume that an element α_i can be chosen in κ_i different ways, where each of those precludes simultaneous choice of another element α_j , $j \in \{1, 2, 3, \dots, \nu\}$, $i \neq j$.

Then the elements: α_1 or α_2 or \dots or α_ν can be chosen in:

$\kappa_1 + \dots + \kappa_\nu$ different ways. (p. 29 ∈ [1])

- **Multiplication Principle:** Assume that an element α_1 can be chosen in κ_1 different ways and for each of those there are κ_2 different ways to choose α_2 etc.

Then the elements: α_1 and α_2 and \dots and α_ν can be chosen in:

$\kappa_1 \kappa_2 \dots \kappa_\nu$ different ways. (p. 29 ∈ [1])

- Assume that we have ν distinct elements $(\alpha_1, \alpha_2, \dots, \alpha_\nu)$, then:

Possible orders:

$$\nu \cdot (\nu - 1) \cdot (\nu - 2) \cdot \dots \cdot 2 \cdot 1 = \nu! \quad \text{(p. 30 ∈ [1])}$$

Orders (permutations) for κ out of ν elements:

$$\nu \cdot (\nu - 1) \cdot \dots \cdot (\nu - \kappa + 1) =$$

$$\frac{\nu!}{(\nu - \kappa)!} = (\nu)_\kappa \quad \text{(thm. 4.1\alpha p. 30 ∈ [1])}$$

Orders for κ out of ν elements with repetition:

$$U(\nu, \kappa) = \nu \cdot \nu \cdot \dots \cdot \nu = \nu^\kappa \quad \text{(thm. 4.1\beta p. 30 ∈ [1])}$$

- **Binomial Coefficient:** Combinations of κ out of ν elements: (thm. 4.1\beta p. 30 ∈ [1])

$$\frac{(\nu)_\kappa}{\kappa!} = \frac{\nu \cdot (\nu - 1) \cdot \dots \cdot (\nu - \kappa + 1)}{\kappa!} = \frac{\nu \cdot (\nu - 1) \cdot \dots \cdot 2 \cdot 1}{\kappa! \cdot 1 \cdot 2 \cdot \dots \cdot (\nu - \kappa)} = \frac{\nu!}{\kappa! (\nu - \kappa)!} = \binom{\nu}{\kappa}$$

- **Combinations of κ out of ν elements with repetition:** (thm. 4.2\beta p. 31

$$\binom{\nu + \kappa - 1}{\kappa} = \binom{\nu - 1 + \kappa}{\kappa} = \binom{\nu - 1 + \kappa}{\nu - 1}$$

- If we place the elements $\alpha_1, \dots, \alpha_\nu$ into r ordered subpopulations with $\kappa_1, \kappa_2, \dots, \kappa_r$ elements respectively, where $\kappa_1 + \kappa_2 + \dots + \kappa_r = \nu$ then:

$$1. \text{ There are } C(\nu, \kappa_1, \kappa_2, \dots, \kappa_{r-1}) = \binom{\nu}{\kappa_1, \kappa_2, \dots, \kappa_{r-1}} = \frac{\nu!}{\kappa_1! \kappa_2! \dots \kappa_{r-1}!}$$

possibilities. (thm. 4.3\alpha p. 32 ∈ [1])

$$2. C(\nu, \kappa_1, \kappa_2, \dots, \kappa_{r-1}) = \binom{\nu}{\kappa_1} \cdot \binom{\nu - \kappa_1}{\kappa_2} \cdot \dots \cdot \binom{\nu - (\kappa_1 + \dots + \kappa_{r-2})}{\kappa_{r-1}}$$

(thm. 4.3 proof p. 32 ∈ [1])

$$3. \text{ We have } \nu \text{ elements of which } \kappa_1 \text{ are alike, } \kappa_2 \text{ are alike, } \dots, \kappa_r \text{ are alike. There are } M(\kappa_1, \kappa_2, \dots, \kappa_r) = C(\nu, \kappa_1, \kappa_2, \dots, \kappa_{r-1}) = \frac{\nu!}{\kappa_1! \kappa_2! \dots \kappa_r!} \text{ permutations. } \quad \text{(thm. 4.3\beta p. 32 ∈ [1])}$$

- **Binomial Development:**

$$(\alpha + \beta)^\nu = \sum_{\kappa=0}^{\nu} \binom{\nu}{\kappa} \alpha^\kappa \beta^{\nu - \kappa}$$

- **Relations of Pascal's Triangle:**

$$\binom{\nu}{\kappa} = \binom{\nu}{\nu - \kappa} = \binom{\nu - 1}{\kappa - 1} + \binom{\nu - 1}{\kappa}$$

- **Stirling's Formula:** $\nu! \cong e^{-\nu} \nu^{\nu + \frac{1}{2}} \sqrt{2\pi}$ (thm. 4.4 p. 33 ∈ [1])

2.1 Combinatorial Principles

- Putting κ *distinct speres* into ν *distinct cells* with *unlimited space*:
 ν^κ *ways*. (ex. 4.1 α p. 36 \in [1])
- Putting κ *distinct speres* into ν *distinct cells* where
cell j contains κ_j spheres and $\kappa_1 + \kappa_2 + \dots + \kappa_\nu = \kappa$:
 $C(\nu, \kappa_1, \kappa_2, \dots, \kappa_{\nu-1}) = \frac{\kappa!}{\kappa_1! \kappa_2! \dots \kappa_{\nu-1}!}$ *ways*. (ex. 4.1 α p. 37 \in [1])
- Putting κ *identical speres* into ν *distinct cells* corresponds to
choosing κ out of ν cells: (ex. 4.1 β p. 37 \in [1])
 - If each cell can *contain 1 sphere*, there are:
 $\binom{\nu}{\kappa}$ *ways*.
 - If each cell has *unlimited space*, there are:
 $\left[\begin{array}{c} \nu \\ \kappa \end{array} \right] = \binom{\nu + \kappa - 1}{\kappa}$ *ways*.
- Ways of choosing κ *ordered elements without repetition* out of ν elements:
 $(\nu)_\kappa$ (ex. 4.2 α p. 28 \in [1])
- Ways of choosing κ *unordered elements without repetition* out of ν el.:
 $\binom{\nu}{\kappa}$ (ex. 4.2 β p. 28 \in [1])
- Ways of choosing κ *ordered elements with repetition* out of ν elements:
 ν^κ (ex. 4.2 γ p. 28 \in [1])
- Ways of choosing κ *unordered elements with repetition* out of ν elements:
 $\left[\begin{array}{c} \nu \\ \kappa \end{array} \right] = \binom{\nu + \kappa - 1}{\kappa}$ (ex. 4.2 δ p. 28 \in [1])

3 Set Theorems

The following holds for *any algebra* and for the usual *set theory*.

- For $A, B, \Gamma \in \mathcal{A}$ we have: (p. 5-10 ∈ [1])
 - Associativity:* $(A \cup B) \cup \Gamma = A \cup (B \cup \Gamma)$
 $(A \cap B) \cap \Gamma = A \cap (B \cap \Gamma)$
 - Distributivity:* $A \cap (B \cup \Gamma) = (A \cap B) \cup (A \cap \Gamma)$
 $A \cup (B \cap \Gamma) = (A \cup B) \cap (A \cup \Gamma)$
 - Commutativity:* $A \cup B = B \cup A$
 $A \cap B = B \cap A$
 - Misc:* $A \cup \emptyset = A$
 $A \cap \Omega = A$
 $A \cup \Omega = \Omega$
 $A \cap \emptyset = \emptyset$
 $\mathbb{C}\Omega = \emptyset$
 $\mathbb{C}\emptyset = \Omega$
 $\mathbb{C}\mathbb{C}A = A$
 $A \cup \mathbb{C}A = \Omega$
 $A \cap \mathbb{C}A = \emptyset$
 - De Morgan:* $\mathbb{C}(\cup_{\nu=1}^{\kappa} A_{\nu}) = \cap_{\nu=1}^{\kappa} (\mathbb{C}A_{\nu})$ (also for infinitely many on a σ -algebra)
 $\mathbb{C}(\cap_{\nu=1}^{\kappa} A_{\nu}) = \cup_{\nu=1}^{\kappa} (\mathbb{C}A_{\nu})$ (also for infinitely many on a σ -algebra)

4 Misc

- **A partition of a sample space** $\Omega \stackrel{def}{=} a$ sequence of possible outcomes $\forall i \in \mathbb{N} : A_i \in \mathcal{A}$ which are *pairwise disjoint* (i.e. $\forall i \neq j : A_i \cap A_j = \emptyset$) and where $\cup_{i=1}^{\infty} A_i = \Omega$. (p. 84 ∈ [1])

References

- [1] Χαραλαμπού Α. Χαραλαμπίδη. *Θεωρία Πιθανοτήτων και Εφαρμογές, Εκδόσεις Συμμετρία Αθήνα 2000.*