Metric Spaces Summary

Ánoq of the Sun, Hardcore Processing *

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1 Metric

- A function $d : M \times M \to \mathbb{R}$ (or *dist*) is called a *metric* or a *distance function* in $M \underset{def}{=}$ the following holds $\forall x, y, z \in M$:
    1. (M1) $d(x,y) \geq 0, d(x,y) = 0 \Leftrightarrow x = y$
    2. (M2) $d(x,y) = d(y,x)$ (symmetry)
    3. (M3) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

  (def. 1.1 [1])
- A *pseudometric* $\underset{def}{=}$ a metric where (M1) is relaxed to $(d(x,y) \geq 0) \wedge (d(x,x) = 0)$. So it can occur that $x \neq y$ and $d(x,y) = 0$ (p. 1.1 [1])

1.1 Default Metrics

- $\mathbb{R}^k$: The *euclidean metric*: $d(x,y) \underset{def}{=} (\sum_{j=a}^{k}(x_j - y_j)^2))^{\frac{1}{2}}$ (p. 1.2 [1])
- $M_1 \times M_2$: *Product metric* of $(M_1, d_1), (M_2, d_2)$: $d(x,y) = \max(d_1(x_1,x_2), d_2(x_1,x_2))$ (p. 4.3 [1])

1.2 Other Well-known Metrics

- For any set: The *discrete metric*: $d(x,y) \underset{def}{=} \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ (ex. 1.7 [1])
- $\mathbb{C}$: $|x - y|$ equals the euclidean metric if $\mathbb{C}$ is identified with $\mathbb{R}^2$ (p. 1.2 [1])
- $\Omega = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$: The *geodetic distance* on the 3D-sphere: $d_\Omega(x,y) \in [0,\pi] \underset{def}{=} \arccos(x \cdot y)$ (1.2 [1])
- All the metrics *induced by the corresponding norms* (p. 1.3 [1])
- For metric d: $d'(x,y) = \min(d(x,y), 1)$ (makes all sets limited) (p. 2.6 [1])

2 Normed Spaces

- A *norm* on a vector space $E = (E, +, \mathbb{L}) \underset{def}{=}$ a function $|| \cdot || : E \to \mathbb{R}$ where the following holds:

    1. (N1) $||x|| \geq 0, ||x|| = 0 \Leftrightarrow x = \overrightarrow{0}$
    2. (N2) $\forall x \in E, \lambda \in \mathbb{L} : ||\lambda x|| = |\lambda| ||x||$
    3. (N3) $\forall x, y \in E : ||x + y|| \leq ||x|| + ||y||$ (triangle inequality)

    (def. 1.2 [1])

- $(E, || \cdot ||)$ is a *normed vector space* (p. 1.3 [1])

- If E is a complex normed vector space: $\forall \theta \in \mathbb{R} : ||e^{i\theta} x|| = ||x||$ (p. 1.3 [1])

- A *seminorm* $\underset{def}{=}$ a norm where (N1) is relaxed to $||x|| \geq 0$. So it can occur that $x \neq 0$ and $||x|| = 0$ (p. 1.3 [1])

- The *metric induced by the norm* $|| \cdot ||$: $d(x, y) \underset{def}{=} ||x - y||$ (p. 1.3 [1])

2.1 Default Norms

- $\mathbb{L}^k$: $||x||_2 \underset{def}{=} (\sum_{j=1}^k |x_j|^2)^{\frac{1}{2}}$ (2-*norm* or *euclidean norm*) (ex. 1.3 [1])

- $\mathcal{B}(M, \mathbb{L}) : ||f||_u \underset{def}{=} \sup\{|f(x)| \mid x \in M\}$ (*uniform norm* or *sup-norm*) (ex. 1.4 [1])

- $C([0,1], \mathbb{R})$: $||f|| = \int_0^1 |f(x)| dx$ (ex. 1.5 [1])

2.2 Other Well-known Norms

- $\mathbb{L}^k$: $||x||_1 \underset{def}{=} \sum_{j=1}^k |x_j|$ (1-*norm*) (ex. 1.3 [1])

- $\mathbb{L}^k$: $||x||_\infty \underset{def}{=} \max(|x_1|, \ldots, |x_k|)$ (∞-*norm* or *max-norm*) (ex. 1.3 [1])

- $\mathcal{F}(\{1, 2, \ldots, k\}, \mathbb{L}) : ||f||_u \equiv \max(|f(1)|, \ldots, |f(k)|)$ (like $|| \cdot ||_\infty$ on $\mathbb{L}^k$) (ex. 1.4 [1])

- $\mathcal{B}(\mathbb{N}, \mathbb{C}) : ||f||_u = \sup\{|f(n)| \mid n \in \mathbb{N}\}$ (ex. 1.4 [1])

- $C([a,b], \mathbb{L}) : ||f||_u = \sup\{|f(x)| \mid x \in [a,b]\}$ (*uniform* norm) (p. 5.4 [1])

- $C([a,b], \mathbb{R}^k) : ||f||_u = \sup\{||f(x)||_\infty \mid x \in [a,b]\}$ (*uniform*) (p. 5.4 [1])

- $C([a,b], \mathbb{R}^k) : ||f||_{u,2} = \sup\{||f(x)||_2 \mid x \in [a,b]\}$ (rem. 5.9 [1])

- $C^k([a,b], \mathbb{L})$: $||f|| = \sum_{j=0}^k ||D^j f||_u$ (thm. 5.10 [1])

3 Spheres in Metric Spaces

Let (M,d) be a metric space:

- $\forall a \in M, r > 0$: The *sphere with center a and radius r*: $K(a,r) \underset{def}{=} \{x \in M \mid d(a,x) < r\}$ (p. 1.6 [1])
- $r_1 < r_2 \Rightarrow K(a,r_1) \subseteq K(a,r_2)$ (p. 1.6 [1])
- $b \in K(a,r) \wedge 0 < s \leq r - d(a,b) \Rightarrow K(b,s) \subseteq K(a,r)$ (sphere lemma) (lemma 1.6 (i) [1])
- $K(a,r) \cap K(b,s) \neq \emptyset \Rightarrow d(a,b) < r + s$ (sphere lemma) (lemma 1.6 (ii) [1])
- A set is *limited* $\underset{def}{=}$ it fits within a sphere (p. 1.7 [1])
- $\forall A \neq \emptyset \subseteq (M,d)$: The *diameter* of A: diam $A \underset{def}{=} \sup\{d(x,y) \mid x,y \in A\}$ (p. 1.6 [1])
- A limited $\Leftrightarrow$ diam $A < \infty$ (p. 1.7 [1])

4 Convergent Sequences

- A *point sequence in a set M* $\underset{def}{=}$ a function $\varphi : \mathbb{N} \to M$. (p. 1.7 [1])
  Notation: If $x_n = \varphi(n)$ then we write: $(x_n)_{n\geq 1}, (x_n)_{n\in\mathbb{N}}$ or (x_n)
- Let $(x_n)_{n\geq 1}$ be a point sequence in the metric space (M,d) and let $a \in M$. $(x_n)_{n\geq 1}$ *converges towards a* $\underset{def}{=} \lim_{n\to\infty} d(x_n,a) = 0$. (def. 1.8 [1])
  We also write $x_n \to a, x_n \to a$ for $n \to \infty, \lim x_n = a, \lim_{n\to\infty} x_n = a$
  Logically: $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n,a) \leq \epsilon$ (p. 1.8 [1]
- If $x_n \to a$ then a is called a *limit point* for the sequence (x_n) (p. 1.8 [1])
- A sequence has *at most one limit point* (rem. 1.9 [1])
- If $x_n \to a$ then any subsequence $(x_{n_p})_{p\geq 1}$ and any suffix sequence $(x_{n+N})_{n\geq 1}$ will also converge towards a (p. 1.8 [1])
- A sequence (x_n) in $\mathbb{R}^k$ (where $x_n = (x_{n1},\ldots,x_{nk})$) converges towards $x = (x_1,\ldots,x_k) \in \mathbb{R}^k \Leftrightarrow \forall j \in \{1,\ldots,k\} : \lim_{n\to\infty} x_{nj} = x_j$ (ex. 1.10 [1])
- Let $(\mathcal{B}(M,\mathbb{L}), ||\cdot||_u)$ be the normed space of limited functions on M. The following holds for a point sequence (f_n) and a function f i $\mathcal{B}(M,\mathbb{L})$: $\lim_{n\to\infty} f_n = f$ in $\mathcal{B}(M,\mathbb{L}) \Leftrightarrow \lim_{n\to\infty} f_n(x) \to f(x)$ uniformly on M (thm. 1.11 [1])

5 Uniform Convergence

- A function sequence *converges uniformly* $\underset{def}{=}$
  $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \forall x \in M : |f(x) - f_n(x)| \leq \epsilon$ (p. 1.9 [1])

6 Topologic Properties in a Metric Space

6.1 Let (M, d) be a metric space and $A \subseteq M$

The interior A° of $A \underset{def}{=}$ the set of all inner points for A	(def. 2.1 [1])
The exterior $(\complement A)^\circ$ of $A \underset{def}{=}$ the set of all outer points for A	(def. 2.1 [1])
The edge ∂A of $A \underset{def}{=}$ the set of all edge points for A	(def. 2.1 [1])
$x \in M$ is an *inner point* $(x \in A^\circ)$ in $A \underset{def}{=} \exists r > 0 : K(x, r) \subseteq A$	(def. 2.1 [1])
$x \in M$ is an *outer point* $(x \in (\complement A)^\circ)$ for $A \underset{def}{=} \exists r > 0 : K(x, r) \cap A = \emptyset$	(def. 2.1 [1])
$x \in M$ is an *edge point* $(x \in \partial A)$ for $A \underset{def}{=} x$ is neither inner nor outer	(def. 2.1 [1])
$x \in \partial A \Leftrightarrow \forall r > 0 : (K(x, r) \cap A \neq \emptyset) \wedge (K(x, r) \cap \complement A \neq \emptyset)$	(p. 2.2 [1])
The *closure of* A: $\overline{A} \underset{def}{=}$ the set of all contact points for A	(def. 2.2 [1])
$x \in M$ is a *contact point* $(x \in \overline{A})$ for $A \underset{def}{=} \forall r > 0 : K(x, r) \cap A \neq \emptyset$	(def. 2.2 [1])
$x \in A$ is an *isolated point* in $A \underset{def}{=} \exists r > 0 : K(x, r) \cap A = \{x\}$	(p. 2.2 [1])
The interior and edge of A are contact points, exterior are not	(p. 2.2 [1])
$M = A^\circ \cup (\complement A)^\circ \cup \partial A$	(p. 2.2 [1])
$\complement A = M \backslash A$	(p. 2.1 [1])
$A^\circ = \overline{A} \backslash \partial A = A \backslash \partial A$	(p. 2.2 [1])
$\partial A = \partial(\complement A) = \overline{A} \backslash A^\circ = \overline{A} \cap \overline{\complement A}$	(p. 2.2 [1])
$\overline{A} = A^\circ \cup \partial A = A \cup \partial A = \complement((\complement A)^\circ)$	(p. 2.2 [1])
$\complement \overline{A} = (\complement A)^\circ$	(p. 2.2 [1])
$A^\circ \subseteq A \subseteq \overline{A}$	(p. 2.2 [1])

6.2 In the Metric Space $\mathbb{R}$

- $\mathbb{Q}^\circ = \emptyset$, $\overline{\mathbb{Q}} = \mathbb{R}$, $\partial \mathbb{Q} = \mathbb{R}$. $\mathbb{Q}$ doesn't have isolated points (ex. 2.3 [1])
- $\mathbb{Z}^\circ = \emptyset$, $\overline{\mathbb{Z}} = \mathbb{Z}$. All points in $\mathbb{Z}$ are isolated (ex. 2.3 [1])

6.3 Open and Closed Sets in a Metric Space (M, d), $A \subseteq M$

- A is *open* $\underset{def}{=} \forall x \in A : x \in A^\circ$, or equivalently $A^\circ = A$ (def. 2.4 [1])
- A *open* $\equiv a \in A \wedge \|u - a\| < \epsilon \Rightarrow u \in U$ (p. 60 [3])
- A is *closed* $\underset{def}{=} \overline{A} = A$ (def. 2.4 [1])
- $\emptyset$ and M are both open and closed (p. 2.2 [1])
- A closed $\Leftrightarrow \complement A$ open (thm. 2.5 [1])
- A open $\Leftrightarrow \complement A$ closed (thm. 2.5 [1])
- The following holds for the system $\mathcal{G} = \mathcal{G}(M)$ of open sets of M:
  1. $\emptyset, M \in \mathcal{G}$
  2. A finite number of sets $G_1, G_2, \ldots, G_n \in \mathcal{G} \Rightarrow G_1 \cap G_2 \cap \cdots \cap G_n \in \mathcal{G}$
  3. An arbitrary family $(G_i)_{i \in I}$ where $\forall i \in I : G_i \in \mathcal{G} \Rightarrow (\bigcup_{i \in I} G_i) \in \mathcal{G}$

  (thm. 2.6 [1])
- The following holds for the system $\mathcal{F} = \mathcal{F}(M)$ of closed sets of M:

1. $\emptyset, M \in \mathcal{F}$
2. A finite number of sets $F_1, F_2, \ldots, F_n \in \mathcal{F} \Rightarrow F_1 \cup F_2 \cup \cdots \cup F_n \in \mathcal{F}$
3. An arbitrary family $(F_i)_{i \in I}$ where $\forall i \in I : F_i \in \mathcal{F} \Rightarrow (\bigcap_{i \in I} F_i) \in \mathcal{F}$

(thm. 2.7 [1])

- A sphere is an open subset (p. 2.4 [1])
- A finite $\Rightarrow A$ is closed and $\forall x \in A : x$ is isolated (p. 2.4 [1])
- A° is open. A° is the largest open subset of A - i.e. $G \subseteq A$ open $\Rightarrow G \subseteq A^\circ$ (thm. 2.8 (a) [1])
- $\overline{A}$ is closed. $\overline{A}$ is the smallest set in M containing A - i.e. $A \subseteq F$ and F closed $\Rightarrow \overline{A} \subseteq F$ (thm. 2.8 (b) [1])
- The exterior of a set $(\complement A)^\circ$ is open (cor. 2.9 [1])
- The edge of a set ∂A is closed (cor. 2.9 [1])

6.4 The following holds for the metric space $(M, d), A \subseteq M$

- $\overline{A}$ are all points $x \in M$ which are *the limit of a convergent sequence* (x_n) in A (i.e. where $\forall n : x_n \in A$) (thm. 2.10 [1])
- A is *dense everywhere* $\underset{def}{=} \overline{A} = M$ (def. 2.11 [1])
- A is dense everywhere $\Leftrightarrow (\forall G \in \mathcal{G} : A \cap G = \emptyset \Rightarrow G = \emptyset)$ (def. 2.11 [1])
- A is dense everywhere $\Leftrightarrow (\forall G \in \mathcal{G} : G \neq \emptyset \Rightarrow A \cap G \neq \emptyset)$ (def. 2.11 [1])
- (M, d) is *separable* $\underset{def}{=}$ there exists a countable subset $A \subseteq M$ which is dense everywhere (def. 2.11 [1])
- $\mathbb{R}^k$ and $\mathbb{C}^k$ are separable because $\mathbb{Q}^k$ and $(\mathbb{Q} + i\mathbb{Q})^k$ are countable and dense everywhere (p. 2.5 [1])
- The following are equivalent for an $x \in M$:
  1. x is a *condensation point* for A
  2. $x \in \overline{A \backslash \{x\}}$ (exc. 2.6 [1])
  3. $\forall r > 0 : K(x, r) \cap A$ contains infinitely many points (exc. 2.6 [1])
  4. $\exists (x_n)$ from $A \backslash \{x\}$ such that $\lim_{n \to \infty} x_n = x$ (exc. 2.6 [1])
  5. There exists a subsequence (x_{n_k}) of (x_n) where $\lim_{n \to \infty} (x_{n_k}) = x$ (thm. 2.6 [2])

6.5 Neighborhoods in (M, d)

- $\mathcal{G}_x \underset{def}{=} \{G \in \mathcal{G} \mid x \in G\}$ (the *system of open sets containing* x) (p. 2.5 [1])
- $U \subseteq M$ is a *neighborhood* of $x \in M \underset{def}{=} x \in U^\circ$ (def. 2.12 [1])
- $\mathcal{U}(x) \underset{def}{=}$ the *system of neighborhoods of* x (def. 2.12 [1])
- $U \subseteq M$ is open $\Leftrightarrow U$ is a neighborhood of all it's points (p. 2.5 [1])
- $U \subseteq M$ is a neighborhood of $x \in M \Leftrightarrow \exists r > 0 : K(x, r) \subseteq U$ (p. 2.5 [1])
- $U \subseteq M$ is a neighborhood of $x \in M \Leftrightarrow \exists G \in \mathcal{G}_x : G \subseteq U$ (p. 2.5 [1])
- $x \in \overline{A} \Leftrightarrow \forall G \in \mathcal{G}_x : A \cap G \neq \emptyset$ (p. 2.6 [1])
- $\lim\limits_{n\to\infty} x_n = x \Leftrightarrow \forall G \in \mathcal{G}_x : \exists N \in \mathbb{N} : x_n \in G$ (p. 2.6 [1])

6.6 Miscellaneous Formulas in (M, d) where $A \subseteq M, B \subseteq M$ and $(A_i)_{i \in I}$ is an arbitrary family where $A_i \in M$

$A \subseteq B$	$\Rightarrow$	$(A^\circ \subseteq B^\circ) \wedge (\overline{A} \subseteq \overline{B})$	(exc. 2.5 [1])
$(A \cap B)^\circ$	$=$	$A^\circ \cap B^\circ$	(exc. 2.5 [1])
$\overline{A \cup B}$	$=$	$\overline{A} \cup \overline{B}$	(exc. 2.5 [1])
$(A \cup B)^\circ$	$\supseteq$	$A^\circ \cup B^\circ$	(exc. 2.5 [1])
$\overline{A \cap B}$	$\subseteq$	$\overline{A} \cap \overline{B}$	(exc. 2.5 [1])
$A^\circ \backslash B^\circ$	$\supseteq$	$(A \backslash B)^\circ$	(exc. 2.5 [1])
$\overline{A} \backslash \overline{B}$	$\subseteq$	$\overline{A \backslash B}$	(exc. 2.5 [1])
$\overline{\bigcup_{i\in I} \overline{A_i}}$	$=$	$\overline{\bigcup_{i \in I} A_i}$	(exc. 2.5 [1])
$(\bigcap_{i \in I} A_i^\circ)^\circ$	$=$	$(\bigcap_{i \in I} A_i)^\circ$	(exc. 2.5 [1])

6.7 Equivalent Metrics in (M, d)

- d_1 and d_2 are *equivalent metrics* on $M \underset{def}{=}$ they define the same system of open sets (def. 2.13 [1])
- All topological properties can be proved by equivalent metrics (p. 2.6 [1])
- Non-topological properties: Sphere, limited set (p. 2.6 [1])
- d_1 and d_2 are equivalent metrics in $M \Leftrightarrow$
  1. $\forall a \in M : \forall r > 0 : \exists s > 0 : K_1(a, r) \supseteq K_2(a, s)$
  2. $\forall a \in M : \forall r > 0 : \exists s > 0 : K_2(a, r) \supseteq K_1(a, s)$

  where $K_i(a, r) = \{x \in M \mid d_i(a, x) < r\}$ (thm. 2.14 [1])

Let E be a vector space over $\mathbb{L}$.

- $|| \cdot ||_1$ and $|| \cdot ||_2$ are *equivalent norms* on $E \underset{def}{=}$ the induced metrics are equivalent (thm. 2.15 [1])
- $|| \cdot ||_1$ and $|| \cdot ||_2$ are equivalent norms on $E \Leftrightarrow$

1. $\exists k > 0 : \forall x \in E : ||x||_1 \leq k||x||_2$
2. $\exists l > 0 : \forall x \in E : ||x||_2 \leq l||x||_1$

(thm. 2.15 [1])

- $||\cdot||_1$ and $||\cdot||_2$ are equivalent norms on $E \Leftrightarrow$
  $\exists c > 0 : \forall x \in E : \frac{1}{c}||x||_1 \leq ||x||_2 \leq c||x||_1$ (rem. 2.16 [1])
- All *norms* on *finite dimensional vector spaces* are *equivalent* (ex. 2.17 [1])

The following holds in $\mathbb{R}^k$:

- $||x||_\infty \leq ||x||_1 \leq k||x||_\infty$ (ex. 2.17 [1])
- $||x||_\infty \leq ||x||_2 \leq \sqrt{k}||x||_\infty$ (ex. 2.17 [1])

6.8 Topological Spaces

- A *topology* on a non-empty set $M \underset{def}{=}$ a system of open sets $\mathcal{G}$ where:
  1. (T1) $\emptyset, M \in \mathcal{G}$
  2. (T2) A finite number of sets $G_1, \ldots, G_n \in \mathcal{G} \Rightarrow G_1 \cap \cdots \cap G_n \in \mathcal{G}$
  3. (T3) An arbitrary family $(G_i)_{i \in I}$ where $\forall i \in I : G_i \in \mathcal{G} \Rightarrow (\bigcup_{i \in I} G_i) \in \mathcal{G}$

  (p. 2.8 [1])
- A *topological space* $(M, \mathcal{G}) \underset{def}{=}$ a set with a topology $\mathcal{G}$. $\mathcal{G}$ are the open sets (p. 2.8 [1])

Let $(M, \mathcal{G})$ be a topological space:

- $\mathcal{G}_x \underset{def}{=} \{G \in \mathcal{G} \mid x \in G\}$ (ex. 2.18 [1])
- $x \in M$ is *inner point* for $A \subseteq M \underset{def}{=} \exists G \in \mathcal{G}_x : G \subseteq A$ (ex. 2.18 [1])
- $x \in M$ is *contact point* for $A \subseteq M \underset{def}{=} \forall G \in \mathcal{G}_x : G \cap A \neq \emptyset$ (ex. 2.18 [1])
- $(M, \mathcal{G})$ has the *Hausdorff property* $\underset{def}{=}$
  $x_1 \neq x_2 \Rightarrow \exists G_1, G_2 \in \mathcal{G} : x_1 \in G_1 \wedge x_2 \in G_2 \wedge G_1 \cap G_2 = \emptyset$
  We can let $G_i = K(x_i, \frac{1}{2}d(x_1, x_2))$ (p. 2.9 [1])

The following metric space formulas also hold in a topological space:

- The formulas regarding $A^\circ, \partial A$ and $\overline{A}$ (p. 2.9 [1])
- The theorems $2.5, 2.7, 2.8$ and corollary 2.9 in [1] (p. 2.9 [1])
- Theorem 2.6 is now a definition (p. 2.9 [1])
- $x \in M$ *is the limit of a convergent sequence* (x_n) in $A \Rightarrow x \in \overline{A}$.
  However, $x \in \overline{A} \not\Rightarrow x$ is *the limit of a convergent sequence* (x_n) in A as in thm. 2.10 [1] (p. 2.9 [1])

7 Continuous Functions

Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces.

- $f : X \to Y$ is *continuous in* $a \in X$ (a is *continuity point for* f) $\underset{def}{=}$ $\forall \epsilon > 0 : \exists \delta > 0 : f(K_X(a, \delta)) \subseteq K_Y(f(a), \epsilon) \equiv$ $\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in X : d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \epsilon$ (p. 3.1 [1])
- $f : X \to Y$ *discontinuous in* $a \in X$ $\underset{def}{=}$ f *not* continuous in a (p. 3.1 [1])
- $f : X \to Y$ is continuous in $a \in X \Leftrightarrow$ (for all sequences (x_n) in X : (x_n) converges toward $a \Rightarrow (f(x_n))$ converges toward $f(a)$) (thm. 3.1 [1])
- $f : X \to Y$ is *continuous* $\underset{def}{=} \forall a \in X : f$ continuous in a (p. 3.2 [1])
- $f : X \to Y$ continuous $\Leftrightarrow \forall G \underset{open}{\subseteq} Y : f^{-1}(G) \subseteq X$ is open (thm. 3.2 [1])
- $f : X \to Y$ continuous $\Leftrightarrow \forall F \underset{closed}{\subseteq} Y : f^{-1}(F) \subseteq X$ closed (thm. 3.3 [1])
- *Coninuity is a topological property* (the 2 theorems above) (p. 3.2 [1])
- f continuous $\Rightarrow \{x \in X \mid f(x) > a\}, \{x \in X \mid a < f(x) < b\}$ are *open in* X (p. 3.3 [1])
- f continuous $\Rightarrow \{x \in X \mid f(x) \leq a\}, \{x \in X \mid a \leq f(x) \leq b\}$ are *closed in* X (p. 3.3 [1])
- f, g continuous, $A \subseteq X$ dense everywhere in X and $\forall x \in A : f(x) = g(x) \Rightarrow f = g$ (thm. 3.4 [1])
- $f : X \to Y, g : Y \to Z$ continuous $\Rightarrow g \circ f$ continuous (thm. 3.5 [1])
- f continuous in $x_0 \in X$ and g continuous in $y_0 = f(x_0) \Rightarrow g \circ f$ continuous in x_0 (rem. 3.6 [1])
- The inverse of a bijective continuous function is generally *not* continuous (warn. 3.7 [1])
- Bijective $f : X \to Y$ is *homeomorphic* $\underset{def}{=} f, f^{-1}$ continuous (def. 3.8 [1])
- Homeomorphisms keep all topological properties (p. 3.4 [1])
- Let $X \underset{open}{\subseteq} \mathbb{R}^k$ with the usual metric d. $f : X \to \mathbb{R}^k$ continuous and injective $\Rightarrow Y = f(X)$ is an open subset of $\mathbb{R}^k$ and f is a homeomorphism of (X, d) into (Y, d) (thm. 3.9 [1])

7.1 Lischitz

Let $(X, d_X), (Y, d_Y)$ be metric spaces.

- $f : X \to Y$ is a *Lipschitz function* with constant $C \underset{def}{=}$ $\forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)$ (p. 3.4 [1])
- f is a *contraction* $\underset{def}{=}$ it is Lipschitz with constant 1 (p. 3.4 [1])

- $f : X \to Y$ is an *isometrism* $\underset{def}{=}$
  $\forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ (p. 3.5 [1])
- Isometrisms are *injective* (p. 3.5 [1])
- The *inverse* of a *surjective isometrism* is also isometric (p. 3.5 [1])
- A *surjective isometrism* is homeomorphic (p. 3.5 [1])
- Let E_1, E_2 be normed real vector spaces. $f : E_1 \to E_2$ is a surjective isometrism and $f(0) = 0 \Rightarrow f$ is linear (thm. 3.10 [1])
- The *j-th projection* $\underset{def}{=} \pi_j(x_1, \ldots, x_k) = x_j$, $j \in \{1, \ldots, k\}$ (p. 3.6 [1])
- π_j is a *contraction* and hence *continuous* (p. 3.6 [1])
- $f : (M, d) \to \mathbb{L}^k$ has k *coordinate functions* $f_j = \pi_j \circ f$ and
  $\forall x \in M : f(x) = (f_1(x), \ldots, f_k(x))$ (p. 3.6 [1])
- f continuous $\Leftrightarrow$ each coordinate function f_j continuous (thm. 3.12 [1])
- $f : (M, d) \to \mathbb{C}$ continuous $\Leftrightarrow Ref, Imf$ continuous (p. 3.7 [1])
- $+, -, \cdot$ on $\mathbb{C} \times \mathbb{C}$ and $\mathbb{R} \times \mathbb{R}$ are continuous (thm. 3.13 [1])
- $(x_1, x_2) \to \frac{x_1}{x_2}$ on $\mathbb{C} \times (\mathbb{C} \backslash \{0\})$ and $\mathbb{R} \times (\mathbb{R} \backslash \{0\})$ continuous (thm. 3.13 [1])

7.2 Let $(E, +, \mathbb{L})$ be a normed space

- $x \to ||x||$ into $\mathbb{R}$ is a contraction and hence continuous. This holds especially for $x \to |x|$ of $\mathbb{R}$ or $\mathbb{C}$ into $\mathbb{R}$ (thm. 3.14 [1])
- A sequence $((x_n, y_n))$ in $\mathbb{R}^2$ or $\mathbb{C}^2$ converges towards $(x, y) \Leftrightarrow (x_n)$ and (y_n) converges towards x and y (p. 3.8 [1])
- $(x_n), (y_n)$ converges towards x, y in $\mathbb{R}$ or $\mathbb{C} \Rightarrow (x_n + y_n), (x_n - y_n), (x_n y_n), (|x_n|)$ converges towards $x + y, x - y, xy, |x|$ (p. 3.8 [1])
- $(x_n), (y_n)$ converges towards x, y in $\mathbb{R}$ or $\mathbb{C}$ and $\forall n : y_n \neq 0$ and $y \neq 0 \Rightarrow \frac{x_n}{y_n}$ converges towards $\frac{x}{y}$ (p. 3.8 [1])

7.3 Continuous real or complex functions

- $f_1, f_2 \in C(M, \mathbb{C}) \Rightarrow f_1 + f_2, f_1 - f_2, f_1 f_2 \in C(M, \mathbb{C})$ (thm. 3.15 [1])
- $(f_1, f_2 \in C(M, \mathbb{C}) \wedge \forall x \in M : f_2(x) \neq 0) \Rightarrow \frac{f_1}{f_2} \in C(M, \mathbb{C})$ (thm. 3.15 [1])
- $f_1, f_2 \in C(M, \mathbb{C}) \Rightarrow |f|, f_1 \vee f_2$ $(= \max\{f_1, f_2\}), f_1 \wedge f_2$ $(= \min\{f_1, f_2\}) \in C(M, \mathbb{C})$ (p. 3.9 [1])
- $f_n : M \to \mathbb{L}$ *converges pointwise* $\underset{def}{=}$
  $\forall x \in M : \forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : |f_n(x) - f(x)| \leq \epsilon$ (p. 3.9 [1])
- $f_n : M \to \mathbb{L}$ *converges uniformly* $\underset{def}{=}$
  $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \forall x \in M : |f_n(x) - f(x)| \leq \epsilon$ (p. 3.9 [1])

- If $\forall n : f_n$ continuous and (f_n) converges *uniformly* towards f then f is continuous (does generally *not* hold for pointwise convergence) (p. 3.9 [1])
- Let (M, d) be a metric space. (f_n) is a sequence of functions which converges uniformly towards f and $\forall n : f_n$ is continuous in $x_0 \in M \Rightarrow$ f continuous in x_0 (thm. 3.16 [1])

8 Metric Space Constructions

8.1 Subspaces, Restriction, Inclusion, Sets Relative to a Space

Let (M, d) be a metric space and (M', d) a metric subspace of (M, d).

- $\forall M' \neq \emptyset \subseteq M : (M', d)$ is a *metric subspace* of (M, d) (d restricted to $M' \times M'$ is the metric) (p. 4.1 [1])
- $\mathcal{G}(M') = \{M' \cap G \mid G \in \mathcal{G}(M)\}$ (*open sets relative* to M') (thm. 4.1 [1])
- $\mathcal{F}(M') = \{M' \cap F \mid F \in \mathcal{F}(M)\}$ (*closed sets relative* to M') (thm. 4.1 [1])
- $\forall A \subseteq M'$: The *closure of A in (M', d)* $\overline{A}^{M'} \underset{def}{=} M' \cap \overline{A}$ (thm. 4.1 [1])
- M' open *in* $M \Leftrightarrow \mathcal{G}(M') \subseteq \mathcal{G}(M)$ (p. 4.2 [1])
- M' open *in* $M \Rightarrow \mathcal{G}(M') = \{G \in \mathcal{G}(M) \mid G \subseteq M'\}$ (p. 4.2 [1])
- The *inclusion function* $i = i_{M',M} : M' \to M \underset{def}{=} i(x) = x$ (p. 4.2 [1])
- $i_{M',M}$ is *isometric* and $\forall G \in \mathcal{G}(M) : i^{-1}(G) = M' \cap G$ (p. 4.2 [1])

In the metric space $\mathbb{R}$ with the usual metric:

- $[0, a[$ open and $[0, a]$ closed relative to $[0, \infty[$ (ex. 4.2 [1])
- $]0, a[$ open and $]0, a]$ closed relative to $]0, \infty[$ (ex. 4.2 [1])

Let $(X, d_X), (Y, d_Y)$ be metric spaces and $\emptyset \neq X' \subseteq X$.

- $f : X \to Y$ continuous in $a \in X' \Leftrightarrow$ the *restriction* $f_{X'} : X' \to Y$ continuous in a, since $f|_{X'} = f \circ i_{X',X}$ (p. 4.2 [1])
- For $f(X) \subseteq Y' \subseteq Y : f : X \to Y'$ continuous in $a \in X \Leftrightarrow f : X \to Y$ continuous in a (p. 4.2 [1])

8.2 Product Spaces

Let $(M_1, d_1), (M_2, d_2)$ be metric spaces, $(M_1 \times M_2, d)$ the metric product space.

- $M_1 \times M_2$ is the *metric product space* of M_1, M_2 with the *product metric* $d(x, y) = \max(d_1(x_1, x_2), d_2(x_1, x_2))$ (p. 4.2 [1])
- $G_1 \in \mathcal{G}(M_1) \wedge G_2 \in \mathcal{G}(M_2) \Rightarrow G_1 \times G_2 \in \mathcal{G}(M_1 \times M_2)$ (thm. 4.3 (a) [1])

- $F_1 \in \mathcal{F}(M_1) \wedge F_2 \in \mathcal{F}(M_2) \Rightarrow F_1 \times F_2 \in \mathcal{F}(M_1 \times M_2)$ (thm. 4.3 (b) [1])
- $\forall A_1 \subseteq M_1, A_2 \subseteq M_2 : (A_1 \times A_2)^\circ = A_1^\circ \times A_2^\circ$ and $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$ (thm. 4.3 (c) [1])
- $(x_n) = (x_{n1}, x_{n2})$ in $M_1 \times M_2$ converges towards $x = (x_1, x_2) \Leftrightarrow$ $(x_{n1}), (x_{n2})$ converges towards $x_1 \in M_1$, $x_2 \in M_2$ (thm. 4.3 (d) [1])
- The *projection functions* $\pi_1(x_1, x_2) = x_1, \pi_2(x_1, x_2) = x_2$ for the product space are *contractions* and hence continuous (p. 4.3 [1])
- $\forall G_1 \subseteq M_1, G_2 \subseteq M_2 : \pi^{-1}(G_1) \cap \pi^{-1}(G_2) = G_1 \times G_2$ (p. 4.4 [1])
- $\forall a \in M_1 : j_a : M_2 \to M_1 \times M_2$ the function $j_a(y) = (a, y)$ is isometric. Similarly for $x \mapsto (x, b)$ (p. 4.4 [1])
- For $f : M_1 \times M_2 \to M_3$ the *slice function* $f \circ j_a = f(a, \cdot) = y \mapsto f(a, y) : M_2 \to M_3$ is continuous. Similarly for $x \mapsto f(x, b)$ (p. 4.4 [1])

8.3 The Space $\mathcal{L}(E, F) \subseteq \mathbf{Hom}(E, F)$

Let E, F be normed vector spaces over $\mathbb{L}$ with norms $||\cdot||$

- $\varphi(x, y) = x + y : E \times E \to E$ and $\psi(\lambda, x) = \lambda x : \mathbb{L} \times E \to E$ are *continuous* (φ is even Lipschitz with constant 2) (thm. 4.4 [1])
- For $T : E \to F$ linear (i.e. $T(x + y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x)$ or just $T \in \mathrm{Hom}(E, G)$) we define $\forall x, y \in E, \lambda \in \mathbb{L}$:
  $||T|| = \sup\{||T(x)|| \mid x \in E, ||x|| \leq 1\} \in [0, \infty]$ (p. 4.5 [1])

Let T be a $k \times k$ matrix and $x \in \mathbb{R}^k$ a column vector. We denote the matrix product with Tx (also gives a column vector) rather than $T(x)$ (rem. 4.5 [1])

- The following are equivalent for a linear map $T : E \to F$:
  1. T continuous in $\vec{0}$
  2. T is Lipschitz (and $||T||$ is the smallest possible Lipschitz constant)
  3. $||T|| < \infty$

  (thm 4.6 [1]
- A linear function is continuous $\Leftrightarrow$ it is limited on the unit sphere $\{x \in E \mid ||x|| \leq 1\}$. We also call this a *limited operator* (rem. 4.7 [1])
- $\mathcal{L}(E, F)$ of continuous linear functions is a *normed vector space* with the *operator norm* $||T|| = \sup\{||Tx|| \mid ||x|| \leq 1\}$ (thm. 4.8 [1])
- $\forall T \in \mathcal{L}(E, F), x \in E$: $||Tx|| \leq ||T|| \, ||x||$ (thm. 4.8 [1])

9 Complete Metric Spaces

Let (M, d) be any metric space.

- (x_n) in M is a *Cauchy sequence* or *fundamental sequence* $\underset{def}{=}$ $\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n, m \geq N : d(x_n, x_m) \leq \varepsilon$ (def. 5.1 [1])
- Any convergent sequence is a Cauchy sequence (p. 5.1 [1])
- (M, d) is *complete* $\underset{def}{=}$ any Cauchy sequence converges (def. 5.2 [1])
- Completeness and Cauchy are *not* topological properties (p. 5.2 [1])
- The completeness of $\mathbb{R}$ equivalent to supremum property of $\mathbb{R}$ (p. 5.1 [1])
- $\mathbb{C}^k$ and $\mathbb{R}^k$ with the usual metrics or d_∞ are complete (p. 5.1 [1])
- $\mathbb{Q}$ is *not* complete with the usual metric (p. 5.1 [1])
- A discrete metric space is complete (exc. 5.3 [1])
- (x_n) Cauchy $\Rightarrow$ (x_n) limited (thm. 2.4 [2])
- (M', d) complete metric subspace of $(M, d) \Rightarrow M'$ closed in M (even when (M, d) is not complete) (rem. 5.4 [1])
- $a \in (M, d)$ is a *condensation point* for a Cauchy sequence $(x_n) \Rightarrow (x_n)$ converges (exc. 5.1 [1], thm. 2.7 [2])

Let (M, d) be a *complete* metric space.

- $\forall M' \neq \emptyset \subseteq M$: (M', d) complete $\Leftrightarrow M'$ closed (thm. 5.3 [1])
- $f : (M, d) \to (Y, d_Y)$ *isometrism* $\Rightarrow (f(X), d_Y)$ *complete* (thm. 5.6 [1])

Misc examples and things which does *not* hold:

- $\arctan : \mathbb{R} \to] - \frac{\pi}{2}, \frac{\pi}{2}[$ is *continuous* and *homeomorphic* (ex. 5.5 [1])
- arctan shows that the image of a complete metric space under a homeomorphism is not necessarily complete ($] - \frac{\pi}{2}, \frac{\pi}{2}[$ not closed) (p. 5.2 [1])
- $\operatorname{dist}(x, y) = |\arctan x - \arctan y|$ defines a metric on $\mathbb{R}$ but $(\operatorname{dist}, \mathbb{R})$ is *not* complete (e.g.: $1, 2, 3, \ldots$ is Cauchy but doesn't converge) (ex. 5.5 [1])

9.1 Banach Spaces

Let E be a normed vector space with *equivalent* norms $|| \cdot ||_1, || \cdot ||_2$.

- (x_n) in E Cauchy w.r.t. $|| \cdot ||_1 \Leftrightarrow (x_n)$ Cauchy w.r.t. $|| \cdot ||_2$ (p. 5.3 [1])
- E complete w.r.t. $|| \cdot ||_1 \Leftrightarrow E$ complete w.r.t. $|| \cdot ||_2$ (p. 5.3 [1])

Banach spaces.

- A *Banach space* $\underset{def}{=}$ *complete normed vector space* (p. 5.3 [1])

- The vector space $\mathcal{B}(M, \mathbb{L})$ with $||\cdot||_u$ is a *Banach space* (thm. 5.7 (a) [1])
- M metric space $\Rightarrow C_b(M, \mathbb{L})$ closed in $\mathcal{B}(M, \mathbb{L})$ and hence a Banach space (thm. 5.7 (b) [1])
- $C([a,b], \mathbb{L})$ is a Banach space with the uniform norm (p. 5.4 [1])
- $C([a,b], \mathbb{R}^k)$ is a Banach space with the uniform norm (thm. 5.8 [1])
- $||\cdot||_u$ *equivalent* to $||f||_{u,2} = \sup\{||f(x)||_2 \mid x \in [a,b]\}$ (rem. 5.9 [1])
- $C^k([a,b], \mathbb{L})$ is a Banach space with $||f|| = \sum_{j=0}^{k} ||D^j f||_u$ (thm. 5.10 [1])
- Let (f_n) be a sequence of C^1 functions where $f_n : [a,b] \to \mathbb{C}$ and:
  1. $\lim\limits_{n\to\infty} f_n'(x) = g(x)$ uniformly for $x \in [a,b]$
  2. $\lim\limits_{n\to\infty} f_n(a) = \alpha$

  Then there exists a C^1 function $f : [a,b] \to \mathbb{C}$ where $\lim\limits_{n\to\infty} f_n(x) = f(x)$ and $\lim\limits_{n\to\infty} f_n'(x) = f'(x)$ (thm. 5.11 [1])
- For normed spaces $E, F, \mathcal{L}(E,F)$: F complete $\Rightarrow \mathcal{L}(E,F)$ complete (thm. 5.12 [1])
- Linear functions $T : E \to \mathbb{L}$ are called *linear forms* or *linear functionals* on E (p. 5.7 [1])
- The set $\mathcal{L}(E, \mathbb{L})$ of linear continuous functions on E is called the *dual space* (E^*) for E (p. 5.7 [1])
- The dual space E^* of continuous linear forms on a normed space E is a Banach space with $||T|| = \sup\{|Tx| \mid ||x|| \le 1\}$ (thm. 5.13 [1])
- A complete metric space $(\widehat{M}, \widehat{d})$ is called a *completion* of a metric space $(M,d) \underset{def}{=}$ there exists an isometrism $\varphi : (M,d) \to (\widehat{M}, \widehat{d})$ where $\varphi(M)$ is dense everywhere in $\widehat{M}$ (p. 5.7 [1])
- *The completion* of a metric space always exist (exc. 5.2 [1])
- *Two completions* of a metric space are always *isometric* (exc. 6.12 [1])
- Any metric space can be seen as a dense everywhere subspace of a complete metric space (p. 5.7 [1])
- Any structure that (M,d) might have also applies to $(\widehat{M}, \widehat{d})$. Hence any normed space space can be seen as an everywhere dense subset of a Banach space (p. 5.7 [1])

10 Compact Sets. Uniform Continuity

Let (M, d) be a metric space.

- $a \in M$ is a *condensation point* for $(x_n) \underset{def}{=} \forall r > 0 : \{n \in \mathbb{N} \mid d(a, x_n) < r\}$ is infinite (def. 6.1 [1])
- A *subsequence* of $(x_n)_{n\in\mathbb{N}}$ in $M \underset{def}{=} (y_p)_{p\in\mathbb{N}}$ in M given by $\forall p \in \mathbb{N} : y_p = x_{n_p}$ where $(n_p)_{p\in\mathbb{N}}$ is a growing sequence in $\mathbb{N}$ (i.e. $p < q \Rightarrow n_p < n_q$). So $(y_p)_{p\in\mathbb{N}} = (x_{n_p})_{p\in\mathbb{N}}$ (def. 6.2 [1])
- (x_n) in (M, d) has condensation point $a \in M \Leftrightarrow$ there exists a subsequece of (x_n) which converges towards a (lemma 6.3 [1])
- $(x_n)_{n\in\mathbb{N}}$ converges in (M, d) towards $a \in M \Rightarrow$ any subsequence of $(x_n)_{n\in\mathbb{N}}$ converges towards $a \in M$. In other words: The limit of a convergent sequence is the only condensation point for the sequence (lemma 6.4 [1])
- $(x_n)_{n\in\mathbb{N}}$ Cauchy with a convergent subsequence $\Rightarrow (x_n)_{n\in\mathbb{N}}$ converges.
  In other words: A Cauchy sequence has at most one condensation point, and it is convergent if it has (lemma 6.4 [1])
- Any *limited real sequence* has *at least one condensation point* (Bolzano-Weierstrass, thm. 6.5 [1])
- $\mathbb{R}$ is complete (and any Cauchy sequence in $\mathbb{R}$ is limited) (p. 6.3 [1])
- $\forall A \subseteq \mathbb{R}^k : A$ limited and closed $\Leftrightarrow$ any sequence in A has a condensation point in A (thm. 6.6 [1])
- $K \subseteq M$ is *compact* $\underset{def}{=}$ any sequence in K has a condensation point in K (def. 6.7 [1])
- (M, d) is a *compact metric space* $\underset{def}{=} M$ is compact (p. 6.5 [1])
- $\emptyset \neq K \subseteq M$ compact subset of $(M, d) \Leftrightarrow (K, d)$ compact (p. 6.5 [1])
- $A \subseteq \mathbb{L}^k$ compact $\Leftrightarrow A$ limited and closed (p. 6.5 [1])
- $A \subseteq M$ compact $\Rightarrow A$ limited and closed (p. 6.5 [1])
- Any *finite* subset of (M, d) is compact (ex. 6.8 [1])
- The only compact sets in a *discrete* metric space are finite set (ex. 6.8 [1])

Let $(X, d_X), (Y, d_Y)$ be a metric spaces.

- $K \underset{compact}{\subseteq} X, f : K \to Y$ continuous $\Rightarrow f(K) \underset{compact}{\subseteq} Y$ (thm. 6.9 [1])
- $\forall \emptyset \neq A \underset{compact}{\subseteq} \mathbb{R} : \sup A \in A \wedge \inf A \in A$ (so A contain max and min numbers) (p. 6.5 [1])
- A continuous real function on $A \underset{compact}{\subseteq} X$ is limited and has both maximum and minimum values (thm. 6.10 [1])

- The metric product space $(X \times Y, d)$ compact $\Leftrightarrow (X, d_X), (Y, d_Y)$ compact (thm. 6.13 [1])
- $f : (X, d_X) \to (Y, d_Y)$ *continuous bijection.* (X, d_X) *compact* $\Rightarrow f$ *homeomorphic* (thm. 6.14 [1])

Let (M, d) be a *compact* metric space.

- $C(M, \mathbb{L})$ is a Banach space with the uniform norm $||f||_u = \sup\{|f(x)| \mid x \in M\} = \max\{|f(x)| \mid x \in M\}$ (cor. 6.11 [1])
- $\forall \neq K \subseteq M : K$ compact $\Leftrightarrow K$ closed (thm. 6.12 [1])

10.1 Equivalence of Norms on Finite Dimensional Vector Spaces

- All norms on $\mathbb{R}^k$ and $\mathbb{C}^k$ are equivalent (thm. 6.15 [1])
- Any finite dimensional normed space is a Banach space (rem. 6.16 [1])
- Finite dimensional subspaces in normed spaces are closed (rem. 6.16 [1])

10.2 Open Coverings

Let (M, d) be a metric space and S be a set.

- A family $(A_i)_{i\in I}$ of subsets of S *covers* $X \subseteq S \underset{def}{=} X \subseteq \bigcup_{i\in I} A_i$ (p. 6.8 [1])
- *Finite* or *numerable covering* $(A_i)_{i\in I} \underset{def}{=} I$ *finite* or *numerable* (p. 6.9 [1])
- $(A_i)_{i\in I}$ can be *thinned out* to $(A_i)_{i\in J} \underset{def}{=} J \subseteq I$ and $(A_i)_{i\in I}, (A_i)_{i\in J}$ covers the same set (p. 6.9 [1])
- An *open covering* $\underset{def}{=}$ a covering where all sets are open (p. 6.9 [1])
- Any *numerable open covering* of a *closed interval* can be *thinned out* to a *finite covering* (p. 6.10 [1])
- $\forall A \subseteq M : A$ *compact* $\Leftrightarrow$ any *open covering* of A can be *thinned out* to a *finite covering* (covering theorem, thm. 6.17 [1])
- Sometimes compactness is defined by the covering ability (any open covering can be thinned out to a finite covering) (rem. 6.18 [1])
- In a topological space the definition: Any sequence in K has a condensation point in K means that K is *sequentially compact* (rem. 6.18 [1])
- Sequential compactness and compactness are not equivalent in a topological space (rem. 6.18 [1])

10.3 Uniform Continuity

Let $(X, d_X), (Y, d_Y)$ be a metric spaces.

- $f : X \to Y$ *uniformly continuous* $\underset{def}{=} \forall \epsilon > 0 : \exists \delta > 0 : \forall x_1, x_2 \in X : d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$ (def. 6.19 [1])
- f uniformly continuous $\Rightarrow f$ pointwise continuous (p. 6.10 [1])
- f Lipschitz $\Rightarrow f$ uniformly continuous (p. 6.10 [1])
- Uniform continuity is *not* a topological property but equivalent norms maintain uniform continuity in normed spaces (p. 6.10 [1])
- $f : (X, d_X) \to (Y, d_Y)$ *continuous*, (X, d_X) *compact* $\Rightarrow f$ *uniformly continuous* (thm. 6.20 [1])
- $A \subseteq X, f : A \to (Y, d_Y)$ uniformly continuous on (A, d_X) and (Y, d_Y) complete $\Rightarrow f$ can only be extended in one way into $\widetilde{f} : \overline{A} \to (Y, d_Y)$ and $\overline{f}$ is even uniformly continuous (6.21 [1])
- Properties of f are inherited to $\widetilde{f}$ (rem. 6.22 [1])
- Let A be a dense subspace of a normed space E and F be a Banach space. $f : A \to F$ continuous linear $\Rightarrow$ the unique continuous extension $\widetilde{f} : E \to F$ of f is linear and $||f|| = ||\widetilde{f}||$ (thm. 6.23 [1])

11 Misc

- $(x_1 + iy_1, \ldots, x_k + iy_k) \mapsto (x_1, y_1, \ldots, x_k, y_k) : \mathbb{C}^k \to \mathbb{R}^{2k}$ is an isometrism (p. 5.1 [1])
- Any set in a discrete metric space is limited (p. 1.7 [1])
- The *distance from a point x to a set $A \neq \emptyset$* $\underset{def}{=}$ $d(x, A) = \inf\{d(x, a) \mid x \in A\}$ (exc. 3.6 [1])
- $x \to d(x, A)$ is a contraction and hence continuous (exc. 3.6 [1])

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