

# Metric Spaces Summary

Ánoq of the Sun, Hardcore Processing \*

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## 1 Metric

- A function  $d : M \times M \rightarrow \mathbb{R}$  (or *dist*) is called a *metric* or a *distance function* in  $M \stackrel{def}{=} \dots$  the following holds  $\forall x, y, z \in M$ :

1. (M1)  $d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y$
2. (M2)  $d(x, y) = d(y, x)$  (symmetry)
3. (M3)  $d(x, y) \leq d(x, z) + d(z, y)$  (triangle inequality)

(def. 1.1 [1])

- A *pseudometric*  $\stackrel{def}{=} \dots$  a metric where (M1) is relaxed to  $(d(x, y) \geq 0) \wedge (d(x, x) = 0)$ . So it can occur that  $x \neq y$  and  $d(x, y) = 0$  (p. 1.1 [1])

### 1.1 Default Metrics

- $\mathbb{R}^k$ : The *euclidean metric*:  $d(x, y) \stackrel{def}{=} (\sum_{j=1}^k (x_j - y_j)^2)^{\frac{1}{2}}$  (p. 1.2 [1])
- $M_1 \times M_2$ : *Product metric* of  $(M_1, d_1), (M_2, d_2)$ :  $d(x, y) = \max(d_1(x_1, x_2), d_2(x_1, x_2))$  (p. 4.3 [1])

### 1.2 Other Well-known Metrics

- For any set: The *discrete metric*:  $d(x, y) \stackrel{def}{=} \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$  (ex. 1.7 [1])
- $\mathbb{C}$ :  $|x - y|$  equals the euclidean metric if  $\mathbb{C}$  is identified with  $\mathbb{R}^2$  (p. 1.2 [1])
- $\Omega = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ : The *geodetic distance* on the 3D-sphere:  $d_\Omega(x, y) \in [0, \pi] \stackrel{def}{=} \arccos(x \cdot y)$  (1.2 [1])
- All the metrics *induced by the corresponding norms* (p. 1.3 [1])
- For metric  $d$ :  $d'(x, y) = \min(d(x, y), 1)$  (makes all sets limited) (p. 2.6 [1])

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## 2 Normed Spaces

- A *norm* on a vector space  $E = (E, +, \mathbb{L}) \stackrel{\text{def}}{=} a$  function  $\|\cdot\| : E \rightarrow \mathbb{R}$  where the following holds:
  1. (N1)  $\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = \vec{0}$
  2. (N2)  $\forall x \in E, \lambda \in \mathbb{L} : \|\lambda x\| = |\lambda| \|x\|$
  3. (N3)  $\forall x, y \in E : \|x + y\| \leq \|x\| + \|y\|$  (triangle inequality)
 (def. 1.2 [1])
- $(E, \|\cdot\|)$  is a *normed vector space* (p. 1.3 [1])
- If  $E$  is a complex normed vector space:  $\forall \theta \in \mathbb{R} : \|e^{i\theta}x\| = \|x\|$  (p. 1.3 [1])
- A *seminorm*  $\stackrel{\text{def}}{=} a$  norm where (N1) is relaxed to  $\|x\| \geq 0$ . So it can occur that  $x \neq 0$  and  $\|x\| = 0$  (p. 1.3 [1])
- The *metric induced by the norm*  $\|\cdot\|$ :  $d(x, y) \stackrel{\text{def}}{=} \|x - y\|$  (p. 1.3 [1])

### 2.1 Default Norms

- $\mathbb{L}^k$ :  $\|x\|_2 \stackrel{\text{def}}{=} (\sum_{j=1}^k |x_j|^2)^{\frac{1}{2}}$  (*2-norm* or *euclidean norm*) (ex. 1.3 [1])
- $\mathcal{B}(M, \mathbb{L})$ :  $\|f\|_u \stackrel{\text{def}}{=} \sup\{|f(x)| \mid x \in M\}$  (*uniform norm* or *sup-norm*) (ex. 1.4 [1])
- $C([0, 1], \mathbb{R})$ :  $\|f\| = \int_0^1 |f(x)| dx$  (ex. 1.5 [1])

### 2.2 Other Well-known Norms

- $\mathbb{L}^k$ :  $\|x\|_1 \stackrel{\text{def}}{=} \sum_{j=1}^k |x_j|$  (*1-norm*) (ex. 1.3 [1])
- $\mathbb{L}^k$ :  $\|x\|_\infty \stackrel{\text{def}}{=} \max(|x_1|, \dots, |x_k|)$  ( *$\infty$ -norm* or *max-norm*) (ex. 1.3 [1])
- $\mathcal{F}(\{1, 2, \dots, k\}, \mathbb{L})$ :  $\|f\|_u \equiv \max(|f(1)|, \dots, |f(k)|)$  (like  $\|\cdot\|_\infty$  on  $\mathbb{L}^k$ ) (ex. 1.4 [1])
- $\mathcal{B}(\mathbb{N}, \mathbb{C})$ :  $\|f\|_u = \sup\{|f(n)| \mid n \in \mathbb{N}\}$  (ex. 1.4 [1])
- $C([a, b], \mathbb{L})$ :  $\|f\|_u = \sup\{|f(x)| \mid x \in [a, b]\}$  (*uniform norm*) (p. 5.4 [1])
- $C([a, b], \mathbb{R}^k)$ :  $\|f\|_u = \sup\{\|f(x)\|_\infty \mid x \in [a, b]\}$  (*uniform*) (p. 5.4 [1])
- $C([a, b], \mathbb{R}^k)$ :  $\|f\|_{u,2} = \sup\{\|f(x)\|_2 \mid x \in [a, b]\}$  (rem. 5.9 [1])
- $C^k([a, b], \mathbb{L})$ :  $\|f\| = \sum_{j=0}^k \|D^j f\|_u$  (thm. 5.10 [1])

### 3 Spheres in Metric Spaces

Let  $(M, d)$  be a metric space:

- $\forall a \in M, r > 0$ : The *sphere with center  $a$  and radius  $r$* :  
 $K(a, r) \stackrel{\text{def}}{=} \{x \in M \mid d(a, x) < r\}$  (p. 1.6 [1])
- $r_1 < r_2 \Rightarrow K(a, r_1) \subseteq K(a, r_2)$  (p. 1.6 [1])
- $b \in K(a, r) \wedge 0 < s \leq r - d(a, b) \Rightarrow K(b, s) \subseteq K(a, r)$  (sphere lemma) (lemma 1.6 (i) [1])
- $K(a, r) \cap K(b, s) \neq \emptyset \Rightarrow d(a, b) < r + s$  (sphere lemma) (lemma 1.6 (ii) [1])
- A set is *limited*  $\stackrel{\text{def}}{=} \text{it fits within a sphere}$  (p. 1.7 [1])
- $\forall A \neq \emptyset \subseteq (M, d)$ : The *diameter* of  $A$ :  $\text{diam } A \stackrel{\text{def}}{=} \sup\{d(x, y) \mid x, y \in A\}$  (p. 1.6 [1])
- $A \text{ limited} \Leftrightarrow \text{diam } A < \infty$  (p. 1.7 [1])

### 4 Convergent Sequences

- A *point sequence in a set  $M$*   $\stackrel{\text{def}}{=} \text{a function } \varphi : \mathbb{N} \rightarrow M$ . (p. 1.7 [1])  
 Notation: If  $x_n = \varphi(n)$  then we write:  $(x_n)_{n \geq 1}, (x_n)_{n \in \mathbb{N}}$  or  $(x_n)$
- Let  $(x_n)_{n \geq 1}$  be a point sequence in the metric space  $(M, d)$  and let  $a \in M$ .  
 $(x_n)_{n \geq 1}$  *converges towards  $a$*   $\stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} d(x_n, a) = 0$ . (def. 1.8 [1])  
 We also write  $x_n \rightarrow a, x_n \rightarrow a \text{ for } n \rightarrow \infty, \lim x_n = a, \lim_{n \rightarrow \infty} x_n = a$   
 Logically:  $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, a) \leq \epsilon$  (p. 1.8 [1])
- If  $x_n \rightarrow a$  then  $a$  is called a *limit point* for the sequence  $(x_n)$  (p. 1.8 [1])
- A sequence has *at most one limit point* (rem. 1.9 [1])
- If  $x_n \rightarrow a$  then any subsequence  $(x_{n_p})_{p \geq 1}$  and any suffix sequence  $(x_{n+N})_{n \geq 1}$  will also converge towards  $a$  (p. 1.8 [1])
- A sequence  $(x_n)$  in  $\mathbb{R}^k$  (where  $x_n = (x_{n1}, \dots, x_{nk})$ ) converges towards  $x = (x_1, \dots, x_k) \in \mathbb{R}^k \Leftrightarrow \forall j \in \{1, \dots, k\} : \lim_{n \rightarrow \infty} x_{nj} = x_j$  (ex. 1.10 [1])
- Let  $(\mathcal{B}(M, \mathbb{L}), \|\cdot\|_u)$  be the normed space of limited functions on  $M$ . The following holds for a point sequence  $(f_n)$  and a function  $f$  in  $\mathcal{B}(M, \mathbb{L})$ :  
 $\lim_{n \rightarrow \infty} f_n = f$  in  $\mathcal{B}(M, \mathbb{L}) \Leftrightarrow \lim_{n \rightarrow \infty} f_n(x) \rightarrow f(x)$  uniformly on  $M$  (thm. 1.11 [1])

### 5 Uniform Convergence

- A function sequence *converges uniformly*  $\stackrel{\text{def}}{=} \forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \forall x \in M : |f(x) - f_n(x)| \leq \epsilon$  (p. 1.9 [1])

## 6 Topologic Properties in a Metric Space

### 6.1 Let $(M, d)$ be a metric space and $A \subseteq M$

The interior $A^\circ$ of $A \stackrel{def}{=} \bar{A}$ the set of all inner points for $A$	(def. 2.1 [1])
The exterior $(\mathbb{C}A)^\circ$ of $A \stackrel{def}{=} \bar{A}$ the set of all outer points for $A$	(def. 2.1 [1])
The edge $\partial A$ of $A \stackrel{def}{=} \bar{A}$ the set of all edge points for $A$	(def. 2.1 [1])
$x \in M$ is an <i>inner point</i> ( $x \in A^\circ$ ) in $A \stackrel{def}{=} \bar{A}$ $\exists r > 0 : K(x, r) \subseteq A$	(def. 2.1 [1])
$x \in M$ is an <i>outer point</i> ( $x \in (\mathbb{C}A)^\circ$ ) for $A \stackrel{def}{=} \bar{A}$ $\exists r > 0 : K(x, r) \cap A = \emptyset$	(def. 2.1 [1])
$x \in M$ is an <i>edge point</i> ( $x \in \partial A$ ) for $A \stackrel{def}{=} \bar{A}$ $x$ is neither inner nor outer	(def. 2.1 [1])
$x \in \partial A \Leftrightarrow \forall r > 0 : (K(x, r) \cap A \neq \emptyset) \wedge (K(x, r) \cap \mathbb{C}A \neq \emptyset)$	(p. 2.2 [1])
The <i>closure of <math>A</math></i> : $\bar{A} \stackrel{def}{=} \bar{A}$ the set of all contact points for $A$	(def. 2.2 [1])
$x \in M$ is a <i>contact point</i> ( $x \in \bar{A}$ ) for $A \stackrel{def}{=} \bar{A}$ $\forall r > 0 : K(x, r) \cap A \neq \emptyset$	(def. 2.2 [1])
$x \in A$ is an <i>isolated point</i> in $A \stackrel{def}{=} \bar{A}$ $\exists r > 0 : K(x, r) \cap A = \{x\}$	(p. 2.2 [1])
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The interior and edge of $A$ are contact points, exterior are not	(p. 2.2 [1])
$M = A^\circ \cup (\mathbb{C}A)^\circ \cup \partial A$	(p. 2.2 [1])
$\mathbb{C}A = M \setminus A$	(p. 2.1 [1])
$A^\circ = \bar{A} \setminus \partial A = A \setminus \partial A$	(p. 2.2 [1])
$\partial A = \partial(\mathbb{C}A) = \bar{A} \setminus A^\circ = \bar{A} \cap \overline{\mathbb{C}A}$	(p. 2.2 [1])
$\bar{A} = A^\circ \cup \partial A = A \cup \partial A = \mathbb{C}((\mathbb{C}A)^\circ)$	(p. 2.2 [1])
$\mathbb{C}\bar{A} = (\mathbb{C}A)^\circ$	(p. 2.2 [1])
$A^\circ \subseteq A \subseteq \bar{A}$	(p. 2.2 [1])

### 6.2 In the Metric Space $\mathbb{R}$

- $\mathbb{Q}^\circ = \emptyset, \overline{\mathbb{Q}} = \mathbb{R}, \partial\mathbb{Q} = \mathbb{R}$ .  $\mathbb{Q}$  doesn't have isolated points (ex. 2.3 [1])
- $\mathbb{Z}^\circ = \emptyset, \overline{\mathbb{Z}} = \mathbb{Z}$ . All points in  $\mathbb{Z}$  are isolated (ex. 2.3 [1])

### 6.3 Open and Closed Sets in a Metric Space $(M, d)$ , $A \subseteq M$

- $A$  is *open*  $\stackrel{def}{=} \forall x \in A : x \in A^\circ$ , or equivalently  $A^\circ = A$  (def. 2.4 [1])
  - $A$  *open*  $\equiv a \in A \wedge \|u - a\| < \epsilon \Rightarrow u \in U$  (p. 60 [3])
  - $A$  is *closed*  $\stackrel{def}{=} \bar{A} = A$  (def. 2.4 [1])
  - $\emptyset$  and  $M$  are both open and closed (p. 2.2 [1])
  - $A$  closed  $\Leftrightarrow \mathbb{C}A$  open (thm. 2.5 [1])
  - $A$  open  $\Leftrightarrow \mathbb{C}A$  closed (thm. 2.5 [1])
  - The following holds for the system  $\mathcal{G} = \mathcal{G}(M)$  of open sets of  $M$ :
    1.  $\emptyset, M \in \mathcal{G}$
    2. A finite number of sets  $G_1, G_2, \dots, G_n \in \mathcal{G} \Rightarrow G_1 \cap G_2 \cap \dots \cap G_n \in \mathcal{G}$
    3. An arbitrary family  $(G_i)_{i \in I}$  where  $\forall i \in I : G_i \in \mathcal{G} \Rightarrow (\bigcup_{i \in I} G_i) \in \mathcal{G}$
- (thm. 2.6 [1])
- The following holds for the system  $\mathcal{F} = \mathcal{F}(M)$  of closed sets of  $M$ :

1.  $\emptyset, M \in \mathcal{F}$
2. A finite number of sets  $F_1, F_2, \dots, F_n \in \mathcal{F} \Rightarrow F_1 \cup F_2 \cup \dots \cup F_n \in \mathcal{F}$
3. An arbitrary family  $(F_i)_{i \in I}$  where  $\forall i \in I : F_i \in \mathcal{F} \Rightarrow (\bigcap_{i \in I} F_i) \in \mathcal{F}$

(thm. 2.7 [1])

- A sphere is an open subset (p. 2.4 [1])
- $A$  finite  $\Rightarrow A$  is closed and  $\forall x \in A : x$  is isolated (p. 2.4 [1])
- $A^\circ$  is open.  $A^\circ$  is the largest open subset of  $A$  - i.e.  $G \subseteq A$  open  $\Rightarrow G \subseteq A^\circ$  (thm. 2.8 (a) [1])
- $\overline{A}$  is closed.  $\overline{A}$  is the smallest set in  $M$  containing  $A$  - i.e.  $A \subseteq F$  and  $F$  closed  $\Rightarrow \overline{A} \subseteq F$  (thm. 2.8 (b) [1])
- The exterior of a set  $(\mathcal{C}A)^\circ$  is open (cor. 2.9 [1])
- The edge of a set  $\partial A$  is closed (cor. 2.9 [1])

#### 6.4 The following holds for the metric space $(M, d), A \subseteq M$

- $\overline{A}$  are all points  $x \in M$  which are *the limit of a convergent sequence*  $(x_n)$  in  $A$  (i.e. where  $\forall n : x_n \in A$ ) (thm. 2.10 [1])
- $A$  is *dense everywhere*  $\stackrel{def}{=} \overline{A} = M$  (def. 2.11 [1])
- $A$  is dense everywhere  $\Leftrightarrow (\forall G \in \mathcal{G} : A \cap G = \emptyset \Rightarrow G = \emptyset)$  (def. 2.11 [1])
- $A$  is dense everywhere  $\Leftrightarrow (\forall G \in \mathcal{G} : G \neq \emptyset \Rightarrow A \cap G \neq \emptyset)$  (def. 2.11 [1])
- $(M, d)$  is *separable*  $\stackrel{def}{=} \overline{A}$  there exists a countable subset  $A \subseteq M$  which is dense everywhere (def. 2.11 [1])
- $\mathbb{R}^k$  and  $\mathbb{C}^k$  are separable because  $\mathbb{Q}^k$  and  $(\mathbb{Q} + i\mathbb{Q})^k$  are countable and dense everywhere (p. 2.5 [1])
- The following are equivalent for an  $x \in M$ :
  1.  $x$  is a *condensation point* for  $A$
  2.  $x \in \overline{A \setminus \{x\}}$  (exc. 2.6 [1])
  3.  $\forall r > 0 : K(x, r) \cap A$  contains infinitely many points (exc. 2.6 [1])
  4.  $\exists (x_n)$  from  $A \setminus \{x\}$  such that  $\lim_{n \rightarrow \infty} x_n = x$  (exc. 2.6 [1])
  5. There exists a subsequence  $(x_{n_k})$  of  $(x_n)$  where  $\lim_{n \rightarrow \infty} (x_{n_k}) = x$  (thm. 2.6 [2])

## 6.5 Neighborhoods in $(M, d)$

- $\mathcal{G}_x \stackrel{def}{=} \{G \in \mathcal{G} \mid x \in G\}$  (the *system of open sets containing  $x$* ) (p. 2.5 [1])
- $U \subseteq M$  is a *neighborhood* of  $x \in M \stackrel{def}{=} x \in U^\circ$  (def. 2.12 [1])
- $\mathcal{U}(x) \stackrel{def}{=} \text{the system of neighborhoods of } x$  (def. 2.12 [1])
- $U \subseteq M$  is open  $\Leftrightarrow U$  is a neighborhood of all its points (p. 2.5 [1])
- $U \subseteq M$  is a neighborhood of  $x \in M \Leftrightarrow \exists r > 0 : K(x, r) \subseteq U$  (p. 2.5 [1])
- $U \subseteq M$  is a neighborhood of  $x \in M \Leftrightarrow \exists G \in \mathcal{G}_x : G \subseteq U$  (p. 2.5 [1])
- $x \in \overline{A} \Leftrightarrow \forall G \in \mathcal{G}_x : A \cap G \neq \emptyset$  (p. 2.6 [1])
- $\lim_{n \rightarrow \infty} x_n = x \Leftrightarrow \forall G \in \mathcal{G}_x : \exists N \in \mathbb{N} : x_n \in G$  (p. 2.6 [1])

## 6.6 Miscellaneous Formulas in $(M, d)$ where $A \subseteq M, B \subseteq M$ and $(A_i)_{i \in I}$ is an arbitrary family where $A_i \in M$

$$\begin{array}{lll}
 A \subseteq B & \Rightarrow & (A^\circ \subseteq B^\circ) \wedge (\overline{A} \subseteq \overline{B}) \quad (\text{exc. 2.5 [1]}) \\
 \overline{(A \cap B)}^\circ & = & A^\circ \cap B^\circ \quad (\text{exc. 2.5 [1]}) \\
 \overline{A \cup B} & = & \overline{A} \cup \overline{B} \quad (\text{exc. 2.5 [1]}) \\
 \overline{(A \cup B)}^\circ & \supseteq & A^\circ \cup B^\circ \quad (\text{exc. 2.5 [1]}) \\
 \overline{A \cap B} & \subseteq & \overline{A} \cap \overline{B} \quad (\text{exc. 2.5 [1]}) \\
 A^\circ \setminus B^\circ & \supseteq & (A \setminus B)^\circ \quad (\text{exc. 2.5 [1]}) \\
 \overline{A \setminus B} & \subseteq & \overline{A \setminus B} \quad (\text{exc. 2.5 [1]}) \\
 \overline{\bigcup_{i \in I} A_i} & = & \overline{\bigcup_{i \in I} A_i} \quad (\text{exc. 2.5 [1]}) \\
 \overline{(\bigcap_{i \in I} A_i)^\circ} & = & (\bigcap_{i \in I} A_i)^\circ \quad (\text{exc. 2.5 [1]})
 \end{array}$$

## 6.7 Equivalent Metrics in $(M, d)$

- $d_1$  and  $d_2$  are *equivalent metrics* on  $M \stackrel{def}{=} \text{they define the same system of open sets}$  (def. 2.13 [1])
- All topological properties can be proved by equivalent metrics (p. 2.6 [1])
- Non-topological properties: Sphere, limited set (p. 2.6 [1])
- $d_1$  and  $d_2$  are equivalent metrics in  $M \Leftrightarrow$ 
  1.  $\forall a \in M : \forall r > 0 : \exists s > 0 : K_1(a, r) \supseteq K_2(a, s)$
  2.  $\forall a \in M : \forall r > 0 : \exists s > 0 : K_2(a, r) \supseteq K_1(a, s)$

where  $K_i(a, r) = \{x \in M \mid d_i(a, x) < r\}$  (thm. 2.14 [1])

Let  $E$  be a vector space over  $\mathbb{L}$ .

- $\|\cdot\|_1$  and  $\|\cdot\|_2$  are *equivalent norms* on  $E \stackrel{def}{=} \text{the induced metrics are equivalent}$  (thm. 2.15 [1])
- $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent norms on  $E \Leftrightarrow$

1.  $\exists k > 0 : \forall x \in E : \|x\|_1 \leq k\|x\|_2$
2.  $\exists l > 0 : \forall x \in E : \|x\|_2 \leq l\|x\|_1$

(thm. 2.15 [1])

- $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent norms on  $E \Leftrightarrow \exists c > 0 : \forall x \in E : \frac{1}{c}\|x\|_1 \leq \|x\|_2 \leq c\|x\|_1$  (rem. 2.16 [1])
- All *norms* on *finite dimensional vector spaces* are *equivalent* (ex. 2.17 [1])

The following holds in  $\mathbb{R}^k$ :

- $\|x\|_\infty \leq \|x\|_1 \leq k\|x\|_\infty$  (ex. 2.17 [1])
- $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{k}\|x\|_\infty$  (ex. 2.17 [1])

## 6.8 Topological Spaces

- A *topology* on a non-empty set  $M \stackrel{def}{=} \mathcal{G}$  a system of open sets  $\mathcal{G}$  where:
  1. (T1)  $\emptyset, M \in \mathcal{G}$
  2. (T2) A finite number of sets  $G_1, \dots, G_n \in \mathcal{G} \Rightarrow G_1 \cap \dots \cap G_n \in \mathcal{G}$
  3. (T3) An arbitrary family  $(G_i)_{i \in I}$  where  $\forall i \in I : G_i \in \mathcal{G} \Rightarrow (\bigcup_{i \in I} G_i) \in \mathcal{G}$

(p. 2.8 [1])

- A *topological space*  $(M, \mathcal{G}) \stackrel{def}{=} \mathcal{G}$  a set with a topology  $\mathcal{G}$ .  $\mathcal{G}$  are the open sets (p. 2.8 [1])

Let  $(M, \mathcal{G})$  be a topological space:

- $\mathcal{G}_x \stackrel{def}{=} \{G \in \mathcal{G} \mid x \in G\}$  (ex. 2.18 [1])
- $x \in M$  is *inner point* for  $A \subseteq M \stackrel{def}{=} \exists G \in \mathcal{G}_x : G \subseteq A$  (ex. 2.18 [1])
- $x \in M$  is *contact point* for  $A \subseteq M \stackrel{def}{=} \forall G \in \mathcal{G}_x : G \cap A \neq \emptyset$  (ex. 2.18 [1])
- $(M, \mathcal{G})$  has the *Hausdorff property*  $\stackrel{def}{=} x_1 \neq x_2 \Rightarrow \exists G_1, G_2 \in \mathcal{G} : x_1 \in G_1 \wedge x_2 \in G_2 \wedge G_1 \cap G_2 = \emptyset$   
We can let  $G_i = K(x_i, \frac{1}{2}d(x_1, x_2))$  (p. 2.9 [1])

The following metric space formulas also hold in a topological space:

- The formulas regarding  $A^\circ, \partial A$  and  $\overline{A}$  (p. 2.9 [1])
- The theorems 2.5, 2.7, 2.8 and corollary 2.9 in [1] (p. 2.9 [1])
- Theorem 2.6 is now a definition (p. 2.9 [1])
- $x \in M$  is the *limit of a convergent sequence*  $(x_n)$  in  $A \Rightarrow x \in \overline{A}$ .  
However,  $x \in \overline{A} \not\Rightarrow x$  is the *limit of a convergent sequence*  $(x_n)$  in  $A$  as in thm. 2.10 [1] (p. 2.9 [1])

## 7 Continuous Functions

Let  $(X, d_X), (Y, d_Y), (Z, d_Z)$  be metric spaces.

- $f : X \rightarrow Y$  is *continuous in*  $a \in X$  ( $a$  is *continuity point for*  $f$ )  $\stackrel{def}{=}$   
 $\forall \epsilon > 0 : \exists \delta > 0 : f(K_X(a, \delta)) \subseteq K_Y(f(a), \epsilon) \equiv$   
 $\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in X : d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \epsilon$  (p. 3.1 [1])
- $f : X \rightarrow Y$  *discontinuous in*  $a \in X$   $\stackrel{def}{=}$   $f$  not continuous in  $a$  (p. 3.1 [1])
- $f : X \rightarrow Y$  is continuous in  $a \in X \Leftrightarrow$  (for all sequences  $(x_n)$  in  $X : (x_n)$  converges toward  $a \Rightarrow (f(x_n))$  converges toward  $f(a)$ ) (thm. 3.1 [1])
- $f : X \rightarrow Y$  is *continuous*  $\stackrel{def}{=}$   $\forall a \in X : f$  continuous in  $a$  (p. 3.2 [1])
- $f : X \rightarrow Y$  continuous  $\Leftrightarrow \forall G \stackrel{\subseteq}{\underset{open}{\subset}} Y : f^{-1}(G) \subseteq X$  is open (thm. 3.2 [1])
- $f : X \rightarrow Y$  continuous  $\Leftrightarrow \forall F \stackrel{\subseteq}{\underset{closed}{\subset}} Y : f^{-1}(F) \subseteq X$  closed (thm. 3.3 [1])
- *Continuity is a topological property* (the 2 theorems above) (p. 3.2 [1])
- $f$  continuous  $\Rightarrow \{x \in X \mid f(x) > a\}, \{x \in X \mid a < f(x) < b\}$  are *open in*  $X$  (p. 3.3 [1])
- $f$  continuous  $\Rightarrow \{x \in X \mid f(x) \leq a\}, \{x \in X \mid a \leq f(x) \leq b\}$  are *closed in*  $X$  (p. 3.3 [1])
- $f, g$  continuous,  $A \subseteq X$  dense everywhere in  $X$  and  $\forall x \in A : f(x) = g(x) \Rightarrow f = g$  (thm. 3.4 [1])
- $f : X \rightarrow Y, g : Y \rightarrow Z$  continuous  $\Rightarrow g \circ f$  continuous (thm. 3.5 [1])
- $f$  continuous in  $x_0 \in X$  and  $g$  continuous in  $y_0 = f(x_0) \Rightarrow g \circ f$  continuous in  $x_0$  (rem. 3.6 [1])
- The inverse of a bijective continuous function is generally *not* continuous (warn. 3.7 [1])
- Bijective  $f : X \rightarrow Y$  is *homeomorphic*  $\stackrel{def}{=}$   $f, f^{-1}$  continuous (def. 3.8 [1])
- Homeomorphisms keep all topological properties (p. 3.4 [1])
- Let  $X \stackrel{\subseteq}{\underset{open}{\subset}} \mathbb{R}^k$  with the usual metric  $d$ .  $f : X \rightarrow \mathbb{R}^k$  continuous and injective  $\Rightarrow Y = f(X)$  is an open subset of  $\mathbb{R}^k$  and  $f$  is a homeomorphism of  $(X, d)$  into  $(Y, d)$  (thm. 3.9 [1])

### 7.1 Lischitz

Let  $(X, d_X), (Y, d_Y)$  be metric spaces.

- $f : X \rightarrow Y$  is a *Lipschitz function* with constant  $C \stackrel{def}{=}$   
 $\forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)$  (p. 3.4 [1])
- $f$  is a *contraction*  $\stackrel{def}{=}$  it is Lipschitz with constant 1 (p. 3.4 [1])

- $f : X \rightarrow Y$  is an *isometrism*  $\stackrel{=}{\text{def}}$   
 $\forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$  (p. 3.5 [1])
- Isometrism are *injective* (p. 3.5 [1])
- The *inverse* of a *surjective isometrism* is also isometric (p. 3.5 [1])
- A *surjective isometrism* is homeomorphic (p. 3.5 [1])
- Let  $E_1, E_2$  be normed real vector spaces.  $f : E_1 \rightarrow E_2$  is a surjective isometrism and  $f(0) = 0 \Rightarrow f$  is linear (thm. 3.10 [1])
- The *j-th projection*  $\stackrel{=}{\text{def}} \pi_j(x_1, \dots, x_k) = x_j, j \in \{1, \dots, k\}$  (p. 3.6 [1])
- $\pi_j$  is a *contraction* and hence *continuous* (p. 3.6 [1])
- $f : (M, d) \rightarrow \mathbb{L}^k$  has *k coordinate functions*  $f_j = \pi_j \circ f$  and  
 $\forall x \in M : f(x) = (f_1(x), \dots, f_k(x))$  (p. 3.6 [1])
- $f$  continuous  $\Leftrightarrow$  each coordinate function  $f_j$  continuous (thm. 3.12 [1])
- $f : (M, d) \rightarrow \mathbb{C}$  continuous  $\Leftrightarrow \text{Ref}, \text{Im}f$  continuous (p. 3.7 [1])
- $+, -, \cdot$  on  $\mathbb{C} \times \mathbb{C}$  and  $\mathbb{R} \times \mathbb{R}$  are continuous (thm. 3.13 [1])
- $(x_1, x_2) \rightarrow \frac{x_1}{x_2}$  on  $\mathbb{C} \times (\mathbb{C} \setminus \{0\})$  and  $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$  continuous (thm. 3.13 [1])

## 7.2 Let $(E, +, \mathbb{L})$ be a normed space

- $x \rightarrow \|x\|$  into  $\mathbb{R}$  is a contraction and hence continuous. This holds especially for  $x \rightarrow |x|$  of  $\mathbb{R}$  or  $\mathbb{C}$  into  $\mathbb{R}$  (thm. 3.14 [1])
- A sequence  $((x_n, y_n))$  in  $\mathbb{R}^2$  or  $\mathbb{C}^2$  converges towards  $(x, y) \Leftrightarrow (x_n)$  and  $(y_n)$  converges towards  $x$  and  $y$  (p. 3.8 [1])
- $(x_n), (y_n)$  converges towards  $x, y$  in  $\mathbb{R}$  or  $\mathbb{C} \Rightarrow (x_n + y_n), (x_n - y_n), (x_n y_n), (|x_n|)$  converges towards  $x + y, x - y, xy, |x|$  (p. 3.8 [1])
- $(x_n), (y_n)$  converges towards  $x, y$  in  $\mathbb{R}$  or  $\mathbb{C}$  and  $\forall n : y_n \neq 0$  and  $y \neq 0 \Rightarrow \frac{x_n}{y_n}$  converges towards  $\frac{x}{y}$  (p. 3.8 [1])

## 7.3 Continuous real or complex functions

- $f_1, f_2 \in C(M, \mathbb{C}) \Rightarrow f_1 + f_2, f_1 - f_2, f_1 f_2 \in C(M, \mathbb{C})$  (thm. 3.15 [1])
- $(f_1, f_2 \in C(M, \mathbb{C}) \wedge \forall x \in M : f_2(x) \neq 0) \Rightarrow \frac{f_1}{f_2} \in C(M, \mathbb{C})$  (thm. 3.15 [1])
- $f_1, f_2 \in C(M, \mathbb{C}) \Rightarrow |f|, f_1 \vee f_2 (= \max\{f_1, f_2\}), f_1 \wedge f_2 (= \min\{f_1, f_2\}) \in C(M, \mathbb{C})$  (p. 3.9 [1])
- $f_n : M \rightarrow \mathbb{L}$  converges *pointwise*  $\stackrel{=}{\text{def}}$   
 $\forall x \in M : \forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : |f_n(x) - f(x)| \leq \epsilon$  (p. 3.9 [1])
- $f_n : M \rightarrow \mathbb{L}$  converges *uniformly*  $\stackrel{=}{\text{def}}$   
 $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \forall x \in M : |f_n(x) - f(x)| \leq \epsilon$  (p. 3.9 [1])

- If  $\forall n : f_n$  continuous and  $(f_n)$  converges *uniformly* towards  $f$  then  $f$  is continuous (does generally *not* hold for pointwise convergence) (p. 3.9 [1])
- Let  $(M, d)$  be a metric space.  $(f_n)$  is a sequence of functions which converges uniformly towards  $f$  and  $\forall n : f_n$  is continuous in  $x_0 \in M \Rightarrow f$  continuous in  $x_0$  (thm. 3.16 [1])

## 8 Metric Space Constructions

### 8.1 Subspaces, Restriction, Inclusion, Sets Relative to a Space

Let  $(M, d)$  be a metric space and  $(M', d)$  a metric subspace of  $(M, d)$ .

- $\forall M' \neq \emptyset \subseteq M : (M', d)$  is a *metric subspace* of  $(M, d)$  ( $d$  restricted to  $M' \times M'$  is the metric) (p. 4.1 [1])
- $\mathcal{G}(M') = \{M' \cap G \mid G \in \mathcal{G}(M)\}$  (*open sets relative to  $M'$* ) (thm. 4.1 [1])
- $\mathcal{F}(M') = \{M' \cap F \mid F \in \mathcal{F}(M)\}$  (*closed sets relative to  $M'$* ) (thm. 4.1 [1])
- $\forall A \subseteq M'$ : The *closure of  $A$  in  $(M', d)$*   $\overline{A}^{M'} \stackrel{def}{=} M' \cap \overline{A}$  (thm. 4.1 [1])
- $M'$  open in  $M \Leftrightarrow \mathcal{G}(M') \subseteq \mathcal{G}(M)$  (p. 4.2 [1])
- $M'$  open in  $M \Rightarrow \mathcal{G}(M') = \{G \in \mathcal{G}(M) \mid G \subseteq M'\}$  (p. 4.2 [1])
- The *inclusion function*  $i = i_{M', M} : M' \rightarrow M \stackrel{def}{=} i(x) = x$  (p. 4.2 [1])
- $i_{M', M}$  is *isometric* and  $\forall G \in \mathcal{G}(M) : i^{-1}(G) = M' \cap G$  (p. 4.2 [1])

In the metric space  $\mathbb{R}$  with the usual metric:

- $[0, a[$  open and  $[0, a]$  closed relative to  $[0, \infty[$  (ex. 4.2 [1])
- $]0, a[$  open and  $]0, a]$  closed relative to  $]0, \infty[$  (ex. 4.2 [1])

Let  $(X, d_X), (Y, d_Y)$  be metric spaces and  $\emptyset \neq X' \subseteq X$ .

- $f : X \rightarrow Y$  continuous in  $a \in X' \Leftrightarrow$  the *restriction*  $f_{X'} : X' \rightarrow Y$  continuous in  $a$ , since  $f|_{X'} = f \circ i_{X', X}$  (p. 4.2 [1])
- For  $f(X) \subseteq Y' \subseteq Y : f : X \rightarrow Y'$  continuous in  $a \in X \Leftrightarrow f : X \rightarrow Y$  continuous in  $a$  (p. 4.2 [1])

### 8.2 Product Spaces

Let  $(M_1, d_1), (M_2, d_2)$  be metric spaces,  $(M_1 \times M_2, d)$  the metric product space.

- $M_1 \times M_2$  is the *metric product space* of  $M_1, M_2$  with the *product metric*  $d(x, y) = \max(d_1(x_1, x_2), d_2(x_1, x_2))$  (p. 4.2 [1])
- $G_1 \in \mathcal{G}(M_1) \wedge G_2 \in \mathcal{G}(M_2) \Rightarrow G_1 \times G_2 \in \mathcal{G}(M_1 \times M_2)$  (thm. 4.3 (a) [1])

- $F_1 \in \mathcal{F}(M_1) \wedge F_2 \in \mathcal{F}(M_2) \Rightarrow F_1 \times F_2 \in \mathcal{F}(M_1 \times M_2)$  (thm. 4.3 (b) [1])
- $\forall A_1 \subseteq M_1, A_2 \subseteq M_2 : (A_1 \times A_2)^\circ = A_1^\circ \times A_2^\circ$  and  $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$  (thm. 4.3 (c) [1])
- $(x_n) = (x_{n1}, x_{n2})$  in  $M_1 \times M_2$  converges towards  $x = (x_1, x_2) \Leftrightarrow (x_{n1}), (x_{n2})$  converges towards  $x_1 \in M_1, x_2 \in M_2$  (thm. 4.3 (d) [1])
- The *projection functions*  $\pi_1(x_1, x_2) = x_1, \pi_2(x_1, x_2) = x_2$  for the product space are *contractions* and hence continuous (p. 4.3 [1])
- $\forall G_1 \subseteq M_1, G_2 \subseteq M_2 : \pi^{-1}(G_1) \cap \pi^{-1}(G_2) = G_1 \times G_2$  (p. 4.4 [1])
- $\forall a \in M_1 : j_a : M_2 \rightarrow M_1 \times M_2$  the function  $j_a(y) = (a, y)$  is isometric. Similarly for  $x \mapsto (x, b)$  (p. 4.4 [1])
- For  $f : M_1 \times M_2 \rightarrow M_3$  the *slice function*  $f \circ j_a = f(a, \cdot) = y \mapsto f(a, y) : M_2 \rightarrow M_3$  is continuous. Similarly for  $x \mapsto f(x, b)$  (p. 4.4 [1])

### 8.3 The Space $\mathcal{L}(E, F) \subseteq \mathbf{Hom}(E, F)$

Let  $E, F$  be normed vector spaces over  $\mathbb{L}$  with norms  $\|\cdot\|$

- $\varphi(x, y) = x + y : E \times E \rightarrow E$  and  $\psi(\lambda, x) = \lambda x : \mathbb{L} \times E \rightarrow E$  are *continuous* ( $\varphi$  is even Lipschitz with constant 2) (thm. 4.4 [1])
- For  $T : E \rightarrow F$  linear (i.e.  $T(x + y) = T(x) + T(y)$  and  $T(\lambda x) = \lambda T(x)$  or just  $T \in \mathbf{Hom}(E, G)$ ) we define  $\forall x, y \in E, \lambda \in \mathbb{L}$ :  
 $\|T\| = \sup\{\|T(x)\| \mid x \in E, \|x\| \leq 1\} \in [0, \infty]$  (p. 4.5 [1])

Let  $T$  be a  $k \times k$  matrix and  $x \in \mathbb{R}^k$  a column vector. We denote the matrix product with  $Tx$  (also gives a column vector) rather than  $T(x)$  (rem. 4.5 [1])

- The following are equivalent for a linear map  $T : E \rightarrow F$ :
  1.  $T$  continuous in  $\vec{0}$
  2.  $T$  is Lipschitz (and  $\|T\|$  is the smallest possible Lipschitz constant)
  3.  $\|T\| < \infty$

(thm 4.6 [1])

- A linear function is continuous  $\Leftrightarrow$  it is limited on the unit sphere  $\{x \in E \mid \|x\| \leq 1\}$ . We also call this a *limited operator* (rem. 4.7 [1])
- $\mathcal{L}(E, F)$  of continuous linear functions is a *normed vector space* with the *operator norm*  $\|T\| = \sup\{\|Tx\| \mid \|x\| \leq 1\}$  (thm. 4.8 [1])
- $\forall T \in \mathcal{L}(E, F), x \in E: \|Tx\| \leq \|T\| \|x\|$  (thm. 4.8 [1])

## 9 Complete Metric Spaces

Let  $(M, d)$  be any metric space.

- $(x_n)$  in  $M$  is a *Cauchy sequence* or *fundamental sequence*  $\stackrel{def}{=}$   
 $\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n, m \geq N : d(x_n, x_m) \leq \varepsilon$  (def. 5.1 [1])
- Any convergent sequence is a Cauchy sequence (p. 5.1 [1])
- $(M, d)$  is *complete*  $\stackrel{def}{=}$  any Cauchy sequence converges (def. 5.2 [1])
- Completeness and Cauchy are *not* topological properties (p. 5.2 [1])
- The completeness of  $\mathbb{R}$  equivalent to supremum property of  $\mathbb{R}$  (p. 5.1 [1])
- $\mathbb{C}^k$  and  $\mathbb{R}^k$  with the usual metrics or  $d_\infty$  are complete (p. 5.1 [1])
- $\mathbb{Q}$  is *not* complete with the usual metric (p. 5.1 [1])
- A discrete metric space is complete (exc. 5.3 [1])
- $(x_n)$  Cauchy  $\Rightarrow (x_n)$  limited (thm. 2.4 [2])
- $(M', d)$  complete metric subspace of  $(M, d) \Rightarrow M'$  closed in  $M$  (even when  $(M, d)$  is not complete) (rem. 5.4 [1])
- $a \in (M, d)$  is a *condensation point* for a Cauchy sequence  $(x_n) \Rightarrow (x_n)$  converges (exc. 5.1 [1], thm. 2.7 [2])

Let  $(M, d)$  be a *complete* metric space.

- $\forall M' \neq \emptyset \subseteq M : (M', d)$  complete  $\Leftrightarrow M'$  closed (thm. 5.3 [1])
- $f : (M, d) \rightarrow (Y, d_Y)$  *isometrism*  $\Rightarrow (f(X), d_Y)$  *complete* (thm. 5.6 [1])

Misc examples and things which does *not* hold:

- $\arctan : \mathbb{R} \rightarrow ] - \frac{\pi}{2}, \frac{\pi}{2}[$  is *continuous* and *homeomorphic* (ex. 5.5 [1])
- $\arctan$  shows that the image of a complete metric space under a homeomorphism is not necessarily complete ( $] - \frac{\pi}{2}, \frac{\pi}{2}[$  not closed) (p. 5.2 [1])
- $\text{dist}(x, y) = |\arctan x - \arctan y|$  defines a metric on  $\mathbb{R}$  but  $(\text{dist}, \mathbb{R})$  is *not* complete (e.g.:  $1, 2, 3, \dots$  is Cauchy but doesn't converge) (ex. 5.5 [1])

### 9.1 Banach Spaces

Let  $E$  be a normed vector space with *equivalent* norms  $\|\cdot\|_1, \|\cdot\|_2$ .

- $(x_n)$  in  $E$  Cauchy w.r.t.  $\|\cdot\|_1 \Leftrightarrow (x_n)$  Cauchy w.r.t.  $\|\cdot\|_2$  (p. 5.3 [1])
- $E$  complete w.r.t.  $\|\cdot\|_1 \Leftrightarrow E$  complete w.r.t.  $\|\cdot\|_2$  (p. 5.3 [1])

Banach spaces.

- A *Banach space*  $\stackrel{def}{=}$  *complete normed vector space* (p. 5.3 [1])

- The vector space  $\mathcal{B}(M, \mathbb{L})$  with  $\|\cdot\|_u$  is a *Banach space* (thm. 5.7 (a) [1])
- $M$  metric space  $\Rightarrow C_b(M, \mathbb{L})$  closed in  $\mathcal{B}(M, \mathbb{L})$  and hence a Banach space (thm. 5.7 (b) [1])
- $C([a, b], \mathbb{L})$  is a Banach space with the uniform norm (p. 5.4 [1])
- $C([a, b], \mathbb{R}^k)$  is a Banach space with the uniform norm (thm. 5.8 [1])
- $\|\cdot\|_u$  equivalent to  $\|f\|_{u,2} = \sup\{\|f(x)\|_2 \mid x \in [a, b]\}$  (rem. 5.9 [1])
- $C^k([a, b], \mathbb{L})$  is a Banach space with  $\|f\| = \sum_{j=0}^k \|D^j f\|_u$  (thm. 5.10 [1])
- Let  $(f_n)$  be a sequence of  $C^1$  functions where  $f_n : [a, b] \rightarrow \mathbb{C}$  and:

1.  $\lim_{n \rightarrow \infty} f'_n(x) = g(x)$  uniformly for  $x \in [a, b]$
2.  $\lim_{n \rightarrow \infty} f_n(a) = \alpha$

Then there exists a  $C^1$  function  $f : [a, b] \rightarrow \mathbb{C}$  where  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  and  $\lim_{n \rightarrow \infty} f'_n(x) = f'(x)$  (thm. 5.11 [1])

- For normed spaces  $E, F, \mathcal{L}(E, F)$ :  $F$  complete  $\Rightarrow \mathcal{L}(E, F)$  complete (thm. 5.12 [1])
- Linear functions  $T : E \rightarrow \mathbb{L}$  are called *linear forms* or *linear functionals* on  $E$  (p. 5.7 [1])
- The set  $\mathcal{L}(E, \mathbb{L})$  of linear continuous functions on  $E$  is called the *dual space*  $(E^*)$  for  $E$  (p. 5.7 [1])
- The dual space  $E^*$  of continuous linear forms on a normed space  $E$  is a Banach space with  $\|T\| = \sup\{|Tx| \mid \|x\| \leq 1\}$  (thm. 5.13 [1])
- A complete metric space  $(\widehat{M}, \widehat{d})$  is called a *completion* of a metric space  $(M, d) \stackrel{\text{def}}{=} \widehat{M}$  there exists an isometrism  $\varphi : (M, d) \rightarrow (\widehat{M}, \widehat{d})$  where  $\varphi(M)$  is dense everywhere in  $\widehat{M}$  (p. 5.7 [1])
- *The completion* of a metric space always exist (exc. 5.2 [1])
- *Two completions* of a metric space are always *isometric* (exc. 6.12 [1])
- Any metric space can be seen as a dense everywhere subspace of a complete metric space (p. 5.7 [1])
- Any structure that  $(M, d)$  might have also applies to  $(\widehat{M}, \widehat{d})$ . Hence any normed space space can be seen as an everywhere dense subset of a Banach space (p. 5.7 [1])

## 10 Compact Sets. Uniform Continuity

Let  $(M, d)$  be a metric space.

- $a \in M$  is a *condensation point* for  $(x_n)_{n \in \mathbb{N}}$   $\stackrel{\text{def}}{=} \forall r > 0 : \{n \in \mathbb{N} \mid d(a, x_n) < r\}$  is infinite (def. 6.1 [1])
- A *subsequence* of  $(x_n)_{n \in \mathbb{N}}$  in  $M$   $\stackrel{\text{def}}{=} (y_p)_{p \in \mathbb{N}}$  in  $M$  given by  $\forall p \in \mathbb{N} : y_p = x_{n_p}$  where  $(n_p)_{p \in \mathbb{N}}$  is a growing sequence in  $\mathbb{N}$  (i.e.  $p < q \Rightarrow n_p < n_q$ ). So  $(y_p)_{p \in \mathbb{N}} = (x_{n_p})_{p \in \mathbb{N}}$  (def. 6.2 [1])
- $(x_n)$  in  $(M, d)$  has condensation point  $a \in M \Leftrightarrow$  there exists a subsequence of  $(x_n)$  which converges towards  $a$  (lemma 6.3 [1])
- $(x_n)_{n \in \mathbb{N}}$  converges in  $(M, d)$  towards  $a \in M \Rightarrow$  any subsequence of  $(x_n)_{n \in \mathbb{N}}$  converges towards  $a \in M$ . In other words: The limit of a convergent sequence is the only condensation point for the sequence (lemma 6.4 [1])
- $(x_n)_{n \in \mathbb{N}}$  Cauchy with a convergent subsequence  $\Rightarrow (x_n)_{n \in \mathbb{N}}$  converges. In other words: A Cauchy sequence has at most one condensation point, and it is convergent if it has (lemma 6.4 [1])
- Any *limited real sequence* has *at least one condensation point* (Bolzano-Weierstrass, thm. 6.5 [1])
- $\mathbb{R}$  is complete (and any Cauchy sequence in  $\mathbb{R}$  is limited) (p. 6.3 [1])
- $\forall A \subseteq \mathbb{R}^k : A$  limited and closed  $\Leftrightarrow$  any sequence in  $A$  has a condensation point in  $A$  (thm. 6.6 [1])
- $K \subseteq M$  is *compact*  $\stackrel{\text{def}}{=} \text{any sequence in } K \text{ has a condensation point in } K$  (def. 6.7 [1])
- $(M, d)$  is a *compact metric space*  $\stackrel{\text{def}}{=} M$  is compact (p. 6.5 [1])
- $\emptyset \neq K \subseteq M$  compact subset of  $(M, d) \Leftrightarrow (K, d)$  compact (p. 6.5 [1])
- $A \subseteq \mathbb{L}^k$  compact  $\Leftrightarrow A$  limited and closed (p. 6.5 [1])
- $A \subseteq M$  compact  $\Rightarrow A$  limited and closed (p. 6.5 [1])
- Any *finite* subset of  $(M, d)$  is compact (ex. 6.8 [1])
- The only compact sets in a *discrete* metric space are finite set (ex. 6.8 [1])

Let  $(X, d_X), (Y, d_Y)$  be a metric spaces.

- $K \stackrel{\subseteq}{\text{compact}} X, f : K \rightarrow Y$  continuous  $\Rightarrow f(K) \stackrel{\subseteq}{\text{compact}} Y$  (thm. 6.9 [1])
- $\forall \emptyset \neq A \stackrel{\subseteq}{\text{compact}} \mathbb{R} : \sup A \in A \wedge \inf A \in A$  (so  $A$  contain max and min numbers) (p. 6.5 [1])
- A continuous real function on  $A \stackrel{\subseteq}{\text{compact}} X$  is limited and has both maximum and minimum values (thm. 6.10 [1])

- The metric product space  $(X \times Y, d)$  compact  $\Leftrightarrow (X, d_X), (Y, d_Y)$  compact (thm. 6.13 [1])
- $f : (X, d_X) \rightarrow (Y, d_Y)$  *continuous bijection*.  $(X, d_X)$  compact  $\Rightarrow f$  *homeomorphic* (thm. 6.14 [1])

Let  $(M, d)$  be a *compact* metric space.

- $C(M, \mathbb{L})$  is a Banach space with the uniform norm  $\|f\|_u = \sup\{|f(x)| \mid x \in M\} = \max\{|f(x)| \mid x \in M\}$  (cor. 6.11 [1])
- $\forall \neq K \subseteq M : K$  compact  $\Leftrightarrow K$  closed (thm. 6.12 [1])

## 10.1 Equivalence of Norms on Finite Dimensional Vector Spaces

- All norms on  $\mathbb{R}^k$  and  $\mathbb{C}^k$  are equivalent (thm. 6.15 [1])
- Any finite dimensional normed space is a Banach space (rem. 6.16 [1])
- Finite dimensional subspaces in normed spaces are closed (rem. 6.16 [1])

## 10.2 Open Coverings

Let  $(M, d)$  be a metric space and  $S$  be a set.

- A family  $(A_i)_{i \in I}$  of subsets of  $S$  *covers*  $X \subseteq S \stackrel{=}{\text{def}} X \subseteq \bigcup_{i \in I} A_i$  (p. 6.8 [1])
- *Finite* or *numerable covering*  $(A_i)_{i \in I} \stackrel{=}{\text{def}} I$  *finite* or *numerable* (p. 6.9 [1])
- $(A_i)_{i \in I}$  can be *thinned out* to  $(A_i)_{i \in J} \stackrel{=}{\text{def}} J \subseteq I$  and  $(A_i)_{i \in I}, (A_i)_{i \in J}$  covers the same set (p. 6.9 [1])
- An *open covering*  $\stackrel{=}{\text{def}}$  a covering where all sets are open (p. 6.9 [1])
- Any *numerable open covering* of a *closed interval* can be *thinned out* to a *finite covering* (p. 6.10 [1])
- $\forall A \subseteq M : A$  *compact*  $\Leftrightarrow$  any *open covering* of  $A$  can be *thinned out* to a *finite covering* (covering theorem, thm. 6.17 [1])
- Sometimes compactness is defined by the covering ability (any open covering can be thinned out to a finite covering) (rem. 6.18 [1])
- In a topological space the definition: Any sequence in  $K$  has a condensation point in  $K$  means that  $K$  is *sequentially compact* (rem. 6.18 [1])
- Sequential compactness and compactness are not equivalent in a topological space (rem. 6.18 [1])

### 10.3 Uniform Continuity

Let  $(X, d_X), (Y, d_Y)$  be a metric spaces.

- $f : X \rightarrow Y$  *uniformly continuous*  $\stackrel{def}{=} \forall \epsilon > 0 : \exists \delta > 0 : \forall x_1, x_2 \in X : d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \epsilon$  (def. 6.19 [1])
- $f$  uniformly continuous  $\Rightarrow f$  pointwise continuous (p. 6.10 [1])
- $f$  Lipschitz  $\Rightarrow f$  uniformly continuous (p. 6.10 [1])
- Uniform continuity is *not* a topological property but equivalent norms maintain uniform continuity in normed spaces (p. 6.10 [1])
- $f : (X, d_X) \rightarrow (Y, d_Y)$  *continuous*,  $(X, d_X)$  *compact*  $\Rightarrow f$  *uniformly continuous* (thm. 6.20 [1])
- $A \subseteq X, f : A \rightarrow (Y, d_Y)$  uniformly continuous on  $(A, d_X)$  and  $(Y, d_Y)$  complete  $\Rightarrow f$  can only be extended in one way into  $\tilde{f} : \bar{A} \rightarrow (Y, d_Y)$  and  $\tilde{f}$  is even uniformly continuous (6.21 [1])
- Properties of  $f$  are inherited to  $\tilde{f}$  (rem. 6.22 [1])
- Let  $A$  be a dense subspace of a normed space  $E$  and  $F$  be a Banach space.  $f : A \rightarrow F$  continuous linear  $\Rightarrow$  the unique continuous extension  $\tilde{f} : E \rightarrow F$  of  $f$  is linear and  $\|\tilde{f}\| = \|f\|$  (thm. 6.23 [1])

### 11 Misc

- $(x_1 + iy_1, \dots, x_k + iy_k) \mapsto (x_1, y_1, \dots, x_k, y_k) : \mathbb{C}^k \rightarrow \mathbb{R}^{2k}$  is an isometrism (p. 5.1 [1])
- Any set in a discrete metric space is limited (p. 1.7 [1])
- The *distance from a point  $x$  to a set  $A \neq \emptyset$*   $\stackrel{def}{=} d(x, A) = \inf\{d(x, a) \mid x \in A\}$  (exc. 3.6 [1])
- $x \rightarrow d(x, A)$  is a contraction and hence continuous (exc. 3.6 [1])

### References

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