Measure Theory Summary

Ánoq of the Sun, Hardcore Processing *

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1 σ-Algebras, Borel Sets, Dynkin Classes

Premises: X ***set***

- $\mathcal{A}$ **is an *algebra*** (on X) $\stackrel{def}{=}$
  $\mathcal{A} \subset \mathcal{P}(X)$ where:

  1. $X \in \mathcal{A}$ (or alternatively: $\emptyset \in \mathcal{A}$ or just $\mathcal{A} \neq \emptyset$)
  2. $(A \in \mathcal{A}) \Rightarrow (\complement A \equiv X \backslash A \in \mathcal{A})$
  3. $(A_1, \ldots, A_\nu \in \mathcal{A}) \Rightarrow (\cap_{\nu \in \{1,\ldots,\nu\}} A_\nu \in \mathcal{A})$ (i.e. *finite intersections*)

  Said differently: A *non-empty family* which is *closed* under *complements* and *finite intersections.* Further more: ⟨def. 1.1 p. 1 ∈ [1]⟩

  - It follows that also *finite unions*, *differences* and $\emptyset$ are in $\mathcal{A}$.
  - 3. Could be also replaced by *finite unions* in the definition.

- $\mathcal{A}$ **is a *σ-algebra*** (on X) $\stackrel{def}{=}$
  $\mathcal{A} \subset \mathcal{P}(X)$ where:

  1. $X \in \mathcal{A}$ (or alternatively: $\emptyset \in \mathcal{A}$ or just $\mathcal{A} \neq \emptyset$)
  2. $(A \in \mathcal{A}) \Rightarrow (\complement A \equiv X \backslash A \in \mathcal{A})$
  3. $(\forall \nu \in \mathbb{N} : A_\nu \in \mathcal{A}) \Rightarrow (\cap_{\nu \in \mathbb{N}} A_\nu \in \mathcal{A})$ (i.e. *countable intersections*)

  Said differently: A *non-empty family* which is *closed* under *complements* and *countable intersections.* Further more: ⟨def. 1.3 p. 1-2 ∈ [1]⟩

  - It follows that also *countable unions*, *differences* and $\emptyset$ are in $\mathcal{A}$.
  - 3. Could be also replaced by *countable unions* in the definition.

- Any *σ-algebra* is also an *algebra.* ⟨rem. 1.4 p. 2 ∈ [1]⟩
- $\{X, \emptyset\}$ is the *smallest σ-algebra* on X.
- $\mathcal{P}(X)$ is the *largest σ-algebra* on X.

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- If $\mathcal{A}$ is an *algebra* on X, then
  $\mathcal{A}$ is *also a σ-algrebra* if *one of the following holds*: ⟨thm. 1.6 p. 3 ∈ [1]⟩

  1. For *all increasing sequences* $\{A_n\}$ in $\mathcal{A}$: $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$
  2. For *all decreasing sequences* $\{A_n\}$ in $\mathcal{A}$: $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$
  3. For *all sequences* $\{A_n\}$ of *pairwise disjoint elems.* in $\mathcal{A}$: $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$

- For a *set* X and a *family* $\mathcal{F}$ of *subsets* of X:

  - There *exists* a *smallest σ-algebra* $\mathcal{A}$ where $\mathcal{F} \subset \mathcal{A}$. ⟨thm. 1.7 p. 3-4 ∈ [1]⟩
  - **The *σ-algebra* $\sigma(\mathcal{F})$ *produced* by** $\mathcal{F} \overset{def}{=}$
    the *smallest σ-algebra* containing $\mathcal{F}$. ⟨thm. 1.7 p. 3-4 ∈ [1]⟩

- Let (X, ρ) be a *metric space* and $\mathcal{T}$ the *family of open subsets* of X, then:

  1. A $G \subset X$ is called a ***G_δ-set*** $\overset{def}{=}$
     There *exists* a *sequence* $\{G_n\}$ of *open subsets* of X such that
     $G = \cap_{n=1}^{\infty} G_n$
  2. An $F \subset X$ is called an ***F_σ-set*** $\overset{def}{=}$
     There *exists* a *sequence* $\{F_n\}$ of *closed subsets* of X such that
     $F = \cup_{n=1}^{\infty} F_n$
  3. A $B \subset X$ is called a ***Borel set*** $\overset{def}{=} B \in \sigma(\mathcal{T})$.
  4. **The *σ-algebra of Borel sets* $\mathcal{B}(X)$ on** $X \overset{def}{=} \sigma(\mathcal{T})$
  5. Clearly $\mathcal{B}(X)$ *contains* the G_δ and the F_σ sets.
  6. *Every closed set* F in X is G_δ. ⟨rem. 1.9 p. 4 ∈ [1]⟩
  7. *Every open set* G in X is F_σ. ⟨rem. 1.9 p. 4 ∈ [1]⟩
  8. A set is $G_\delta \Leftrightarrow$ it's *complement* is F_σ. ⟨p. 4 ∈ [1]⟩

- A *family* $\mathcal{D}$ of *subsets* of X is a ***Dynkin class*** on $X \overset{def}{=}$

  1. $X \in \mathcal{D}$
  2. $(A, B \in \mathcal{D} \wedge A \subset B) \Rightarrow B \backslash A \in \mathcal{D}$
  3. $\{A_n\}$ *increasing sequence* in $\mathcal{D} \Rightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{D}$

  ⟨def. 1.12 p. 5 ∈ [1]⟩

- The ***Dynkin class*** $\delta(\Delta)$ ***produced** by* the *family* Δ of *subsets* of $X \overset{def}{=}$
  the *smallest* Dynkin class *containing* Δ. ⟨p. 5 ∈ [1]⟩

- *Every σ-algebra* is a *Dynkin class* (i.e. $\forall \Delta : \sigma(\Delta) \subset \delta(\Delta)$). ⟨p. 5 ∈ [1]⟩

- For *family* Δ of *subsets* of X:
  If Δ *closed under finite intersections* then $\delta(\Delta) = \sigma(\Delta)$. ⟨thm. 1.13 p. 6 ∈ [1]⟩

- For the *family* $\mathcal{F}$ of *closed subsets* of $\mathbb{R}^k$, where $k \in \mathbb{N}$ and for:
  $\Delta_1 = \{\prod_{i=1}^{k}]-\infty, \beta_i] \mid \forall i \in \{1, \ldots, k\} : \beta_i \in \mathbb{R}\}$
  $\Delta_2 = \{\prod_{i=1}^{k}]\alpha_i, \beta_i] \mid \forall i \in \{1, \ldots, k\} : \alpha_i, \beta_i \in \mathbb{R}, \alpha_i < \beta_i\}$
  $\Delta_3 = \{\prod_{i=1}^{k}]\alpha_i, \beta_i[\mid \forall i \in \{1, \ldots, k\} : \alpha_i, \beta_i \in \mathbb{R}, \alpha_i < \beta_i\}$
  we have that: $\mathcal{B}(\mathbb{R}^k) = \sigma(\mathcal{F}) = \sigma(\Delta_1) = \sigma(\Delta_2) = \sigma(\Delta_3)$. ⟨thm. 1.11 p. 5 ∈ [1]⟩
  and: $\mathcal{B}(\mathbb{R}) = \delta(\mathcal{F}) = \delta(\Delta_1) = \delta(\Delta_2) = \delta(\Delta_3)$. ⟨rem. 1.14 p. 6 ∈ [1]⟩

2 Measures

Premises: X ***set***

- For *σ-algebra* $\mathcal{A}$ on X: The *set function* $\mu : \mathcal{A} \to [0, \infty]$ is a ***(countably additive / σ-additive) measure*** $\stackrel{def}{=}$ ⟨def. 2.1 p. 10 ∈ [1]⟩
  1. $\mu(\emptyset) = 0$
  2. If $\{A_n\}$ is a *sequence* of *pairwise disjoint* elements of $\mathcal{A}$ then $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$. (*countable additivity / σ-additivity*)
- For *algebra* $\mathcal{A}$ on X: The *set function* $\mu : \mathcal{A} \to [0, \infty]$ is a ***finitely additive measure*** $\stackrel{def}{=}$ ⟨p. 10 ∈ [1]⟩
  1. $\mu(\emptyset) = 0$
  2. If $\{A_i\}$ is a *finite sequence* of *pairwise disjoint* elements of $\mathcal{A}$ then $\mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$. (*finite additivity*)
- ***Measurable space*** $\stackrel{def}{=}$ $(X, \mathcal{A})$ where $\mathcal{A}$ is a *σ-algebra* on X. ⟨p. 10 ∈ [1]⟩
- ***Measure space*** $\stackrel{def}{=}$ $(X, \mathcal{A}, \mu)$ where $(X, \mathcal{A})$ is a *measurable space* with *measure* μ. ⟨p. 10 ∈ [1]⟩
- *Any measure* in a *measurable space* is also a *finitely additive measure.* ⟨p. 10 ∈ [1]⟩
- Examples of measures:
  - ***Arithmetic measure*** $\stackrel{def}{=}$ ⟨ex. 2.2 p. 11 ∈ [1]⟩
    $\mu(A) = \begin{cases} n & \text{, if } A \text{ has } n \text{ elements} \\ \infty & \text{, if } A \text{ is an } \textit{infinite set} \end{cases}$.
  - ***Dirac measure*** δ_x at the *point* x $\stackrel{def}{=}$ ⟨ex. 2.2 p. 11 ∈ [1]⟩
    $\delta_x(A) = \begin{cases} 1 & \text{, if } x \in A \\ 0 & \text{, if } x \notin A \end{cases}$.
  - If μ, ν *measures* and $\alpha \in \mathbb{R}, \alpha \geq 0$, then $\mu + \nu$ and $\alpha \cdot \mu$ are *also measures.* ⟨p. 11 ∈ [1]⟩

- For *measure space* $(X, \mathcal{A}, \mu)$:
  - If $A, B \in \mathcal{A}$ and $A \subset B$, then $\mu(A) \leq \mu(B)$. (i.e. μ is a *monotonous set function*) ⟨thm. 2.3 p. 11 ∈ [1]⟩
  - If $A, B \in \mathcal{A}$ and $A \subset B$ and $\mu(A) < \infty$, then $\mu(B \backslash A) = \mu(B) - \mu(A)$. ⟨thm. 2.3 p. 11 ∈ [1]⟩
  - If $\{A_n\}$ is a *sequence* of *elements* of $\mathcal{A}$, then $\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.
    (i.e. μ is a *countably subadditive set function*) ⟨thm. 2.4 p. 11 ∈ [1]⟩
  - If $\{A_n\}$ *increasing sequence* in $\mathcal{A}$ then $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n) = \sup_{n \to \infty} \mu(A_n)$. ⟨thm. 2.5 (i) p. 12 ∈ [1]⟩
  - For $\{A_n\}$ *decreasing sequence* in $\mathcal{A}$ and $\mu(A_1) < \infty$ then $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n) = \inf_{n \to \infty} \mu(A_n)$. ⟨thm. 2.5 (ii) p. 12 ∈ [1]⟩
    Warning: Does *not hold* if $\mu(A_1) = \infty$. E.g.: In $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the *arithmetic measure* and $A_n = \{n, n+1, \ldots\}$.

- For *measurable space* $(X, \mathcal{A})$ and μ *finitly additive measure* in $(X, \mathcal{A})$.
  μ is a *measure* if *one of the following* holds: ⟨thm. 2.6 p. 13 ∈ [1]⟩
  1. For every *increasing sequence* $\{A_n\}$ in $\mathcal{A}$ it holds that: $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n)$.
  2. For every *decreasing sequence* $\{A_n\}$ in $\mathcal{A}$ with $\cap_{n=1}^{\infty} A_n = \emptyset$ it holds: $\lim_{n \to \infty} \mu(A_n) = 0$.

- For *measure space* $(X, \mathcal{A}, \mu)$: ⟨def. 2.7 p. 13 ∈ [1]⟩
  - μ **is *limited*** $\stackrel{def}{=} \mu(X) < \infty$.
    $(X, \mathcal{A}, \mu)$ is called a *limited measure space*.
  - μ **is a *probability measure*** $\stackrel{def}{=} \mu(X) = 1$.
    $(X, \mathcal{A}, \mu)$ is called a *probability measure space*.
  - μ **is σ-*limited*** $\stackrel{def}{=}$ there *exists* a *sequence* $\{A_n\}$ in $\mathcal{A}$ such that $\cup_{n=1}^{\infty} A_n = X$ and $\forall n : \mu(A_n) < \infty$.
    $(X, \mathcal{A}, \mu)$ is called a *σ-limited measure space*.

- For *measurable space* $(X, \mathcal{A})$ and a *family* Δ of *subsets* of X which is *closed under finite intersections* and where $\sigma(\Delta) = \mathcal{A}$, and μ, ν are *measures* in $(X, \mathcal{A})$ such that $\forall D \in \Delta : \mu(D) = \nu(D)$.
  If *one of the following holds* then $\mu = \nu$: ⟨thm. 2.8 p. 14 ∈ [1]⟩
  1. $\mu(X) = \nu(X) < \infty$. (i.e. μ, ν are *limited*)
  2. There *exists* an *increasing sequence* $\{D_n\}$ in Δ such that $\cup_{n=1}^{\infty} D_n = X$ and $\forall D_n : \mu(D_n) = \nu(D_n) < \infty$. (i.e. μ, ν σ-*limited*)

2.1 μ-Zero Sets, μ-Measurable Sets, Completions

Premises: X ***set***, $(X, \mathcal{A}, \mu)$ ***measure space***.

- $N \subset X$ **is μ-zero** $\overset{def}{=}$ ⟨def. 2.9 p. 15 ∈ [1]⟩
  there *exists* $A \in \mathcal{A}$ such that $\mu(A) = 0$ and $N \subset A$.
- $(X, \mathcal{A}, \mu)$ **is a *complete measure space* and μ a *complete measure***
  $\overset{def}{=}$ every *μ-zero set* N *belongs* to $\mathcal{A}$. (and thus $\mu(N) = 0$) ⟨def. 2.9 p. 15 ∈ [1]⟩
- **The *completion* / *μ-measurable sets* $\mathcal{A}_\mu$ of $\mathcal{A}$ w.r.t. μ** $\overset{def}{=}$
  the *family* $\mathcal{A}_\mu$ of all $A \subset X$ such that:
  $\exists E, F \in \mathcal{A} : E \subset A \subset F \wedge \mu(F \backslash E) = 0.$ ⟨def. 2.9 p. 15 ∈ [1]⟩
- **The *completion* $\overline{\mu} : \mathcal{A}_\mu \to [0, \infty]$ of μ** $\overset{def}{=}$
  $\overline{\mu}(A) = \mu(E)$ where $E \in \mathcal{A}$ is such that:
  $\exists F \in \mathcal{A} : E \subset A \subset F \wedge \mu(F \backslash E) = 0.$ ⟨def. 2.9 p. 15 ∈ [1]⟩
  Note: $\mu(E) = \sup\{\mu(B) \mid B \in \mathcal{A}, B \subset A\}$.
- $(X, \mathcal{A}_\mu, \overline{\mu})$ **is the *completion*** of $(X, \mathcal{A}, \mu)$. ⟨def. 2.9 p. 15 ∈ [1]⟩
- $\mathcal{A}_\mu$ is a *σ-algebra* and $\mathcal{A}_\mu \supset \mathcal{A}$. ⟨thm. 2.10 p. 15 ∈ [1]⟩
- $\overline{\mu}$ is a *complete measure* and $\overline{\mu}|_{\mathcal{A}} = \mu$. ⟨thm. 2.10 p. 15 ∈ [1]⟩
- $\overline{\mu}$ is the *unique measure* such that $\overline{\mu}|_{\mathcal{A}} = \mu$. ⟨thm. 2.10 p. 15 ∈ [1]⟩
- μ is *complete* $\Leftrightarrow \mathcal{A} = \mathcal{A}_\mu$. ⟨thm. 2.10 p. 15 ∈ [1]⟩

3 Exterior Measures

Premises: X **is a *set*.**

- **A set function $\phi : \mathcal{P}(X) \to [0, \infty]$ is an *exterior measure* on X** $\overset{def}{=}$
  1. $\phi(\emptyset) = 0$.
  2. If $A \subset B \subset X$, then $\phi(A) \leq \phi(B)$. (monotony)
  3. If $\{A_n\}$ is a *sequence* of *subsets* of X then $\phi(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \phi(A_n)$. (countable subadditivity / σ-subadditivity)

  ⟨def. 3.1 p. 19 ∈ [1]⟩
- **The *exterior Lebesgue measure* $\lambda^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ on $\mathbb{R}$** $\overset{def}{=} \lambda^*(A) = \inf\{\sum_{n=1}^{\infty}(\beta_n - \alpha_n) \mid A \subset \sum_{n=1}^{\infty}]\alpha_n, \beta_n[, \alpha_m, \beta_m \in \mathbb{R}, \alpha_m < \alpha_m, \forall m \in \mathbb{N}\}$.
  ⟨def. 3.3 p. 19 ∈ [1]⟩
- λ^* is an *exterior measure* and:
  $\forall \alpha, \beta \in \mathbb{R}, \alpha < \beta : \lambda^*([\alpha, \beta]) = \lambda(]\alpha, \beta]) = \lambda([\alpha, \beta[) = \lambda(]\alpha, \beta[) = \beta - \alpha$.
  $\lambda^*(I) = \infty$ if I is an *unlimited interval.* ⟨thm. 3.4 p. 20 ∈ [1]⟩
- **The *exterior Lebesgue measure* $\lambda_k^* : \mathcal{P}(\mathbb{R}^k) \to [0, \infty]$ on $\mathbb{R}^k$** $\overset{def}{=} \lambda_k^*(A) = \inf\{\sum_{n=1}^{\infty} v(I_n) \mid A \subset \sum_{n=1}^{\infty} I_n, I_m \underset{open\ interval}{\subset} \mathbb{R}^k, \forall m \in \mathbb{N}\}$, where $v(I) = (\beta_1 - \alpha_1)(\beta_2 - \alpha_2) \cdots (\beta_k - \alpha_k)$ is the *volume* of I. ⟨def. 3.5 p. 22 ∈ [1]⟩
- λ_k^* is an *exterior measure* and:
  $\forall I \underset{interval}{\subset} \mathbb{R}^k : \lambda_k^*(I) = v(I)$. ⟨thm. 3.7 p. 24 ∈ [1]⟩
- The family Δ of all *subsets* of $\mathbb{R}^k$ which can be written as a *finite union* of *disjoint intervals* of $\mathbb{R}^k$ is an *algebra* on $\mathbb{R}^k$.
  ⟨lemma 3.6 (i) p. 22 ∈ [1]⟩
- For *pairwise disjoint intervals* I_i in $\mathbb{R}^k$, $I = \cup_{i=1}^{\infty} I_i$:
  If J is an *interval* such that $I \subset J$ then: $\sum_{i=1}^{\infty} v(I_i) \leq v(J)$.
  If I is an *interval* then: $\sum_{i=1}^{\infty} v(I_i) = v(I)$. ⟨lemma 3.6 (ii) p. 22 ∈ [1]⟩
- For *exterior measure* $\phi : \mathcal{P}(X) \to [0, \infty]$ on X:
  - $B \subset X$ **is *ϕ-measurable*** $\overset{def}{=} \forall A \subset X : \phi(A) = \phi(A \cap B) + \phi(A \backslash B)$.
    It is enough to show (because ϕ is an *exterior measure*):
    $\forall A \subset X, \phi(A) < \infty : \phi(A) \geq \phi(A \cap B) + \phi(A \backslash B)$. ⟨def. 3.8 p. 25 ∈ [1]⟩
  - **The *family* of *ϕ-measurable sets* is symbolized by $\mathcal{M}_\phi$.**
    ⟨def 3.8 p. 25 ∈ [1]⟩
  - *Every set* $B \subset X$ with $\phi(B) = 0$ is *ϕ-measurable.* ⟨p. 25 ∈ [1]⟩
  - *Καραθεοδωρή*: ⟨thm. 3.9 p. 25 ∈ [1]⟩
    $\mathcal{M}_\phi$ is a *σ-algebra* on X and $\phi|_{\mathcal{M}_\phi}$ is a *complete measure.*
- $\mathcal{M}_{\lambda_k^*}$ **is the *Lebesgue measurable sets*.** ⟨p. 26 ∈ [1]⟩
- $\mathcal{B}(\mathbb{R}^k) \subset \mathcal{M}_{\lambda_k^*}$. ⟨thm. 3.10 p. 26 ∈ [1]⟩
- **The *Lebesgue measure* λ_k** $\overset{def}{=}$ ⟨def. 3.11 p. 27-28 ∈ [1]⟩
  $\lambda_k^*|_{\mathcal{M}_{\lambda_k^*}}$ or sometimes the more restricted $\lambda_k^*|_{\mathcal{B}(\mathbb{R}^k)}$.

- For any $A \subset \mathbb{R}^k$: $\lambda_k^*(A) =$
$\inf\{\lambda_k(B) \mid B \in \mathcal{B}(\mathbb{R}^k), B \supset A\} =$
$\inf\{\lambda_k(U) \mid U \underset{open}{\subset} \mathbb{R}^k, U \supset A\}$. ⟨thm. 3.12 p. 28 ∈ [1]⟩

- For *measure space* $(X, \mathcal{A}, \mu)$:

  - **The *exterior measure* μ^* w.r.t. μ** $\overset{def}{=}$
  $\forall A \subset X : \mu^*(A) = \inf\{\mu(B) \mid B \in \mathcal{A} \wedge B \supset A\}$. ⟨def. 3.13 p. 28 ∈ [1]⟩
  - **The *interior measure* μ_* w.r.t. μ** $\overset{def}{=}$
  $\forall A \subset X : \mu_*(A) = \sup\{\mu(B) \mid B \in \mathcal{A} \wedge B \subset A\}$. ⟨def. 3.13 p. 28 ∈ [1]⟩
  - $\forall A \subset X : \exists B \in \mathcal{A} : A \subset B \wedge \mu^*(A) = \mu(B)$. ⟨thm. 3.14 (i) p. 29 ∈ [1]⟩
  - μ^* is an *exterior measure* on X. ⟨thm. 3.14 (ii) p. 29 ∈ [1]⟩
  - $\mathcal{A}_\mu \subset \mathcal{M}_{\mu^*}$ and $\mu^*|_{\mathcal{A}_\mu} = \overline{\mu}$. ⟨thm. 3.14 (iii) p. 29 ∈ [1]⟩
  - If μ is *σ-limited*, then $\mathcal{A}_\mu = \mathcal{M}_{\mu^*}$ and $\mu^*|_{\mathcal{M}_{\mu^*}} = \overline{\mu}$.
  ⟨thm. 3.14 (iv) p. 29 ∈ [1]⟩

- If λ is the *Lebesgue measure* on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ and
$\lambda^* : \mathcal{P}(\mathbb{R}^k) \to [0, \infty]$ with $\lambda^*(A)$ being the *exterior measure* of A w.r.t. λ,
then λ^* *conincides* with the *exterior Lebesgue measure* on $\mathbb{R}^k$.
⟨rem. 3.15 p. 30 ∈ [1]⟩

- The *Lebesgue measure* on $(\mathbb{R}^k, \mathcal{M}_{\lambda^*})$ is the
completion of the *Lebesgue measure* on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$. ⟨thm. 3.16 p. 30 ∈ [1]⟩

- For *measure space* $(X, \mathcal{A}, \mu)$ and $A \subset X$ with $\mu^*(A) < \infty$:
$A \in \mathcal{A}_\mu \Leftrightarrow \mu_*(A) = \mu^*(A)$. ⟨thm. 3.17 p. 30 ∈ [1]⟩

3.1 Basic Properties of the Lebesgue Measure

- For *metric space* X and σ*-algebra* $\mathcal{A}$ on X with $\mathcal{A} \supset \mathcal{B}(X)$ and μ *measure* on $(X, \mathcal{A})$: μ **is *normal* / *regular?*** $\stackrel{def}{=}$ ⟨def. 4.1 p. 37 ∈ [1]⟩

  1. $\forall K \underset{compact}{\subset} X : \mu(K) < \infty$
  2. $\forall A \in \mathcal{A} : \mu(A) = \inf\{\mu(U) \mid U \underset{open}{\subset} X, U \supset A\}$
     (*exterior normality / regularity?*)
  3. $\forall U \underset{open}{\subset} X : \mu(U) = \sup\{\mu(K) \mid K \underset{compact}{\subset} X, K \subset U\}$
     (*interior normality / regularity?*)

- The *Lebesgue measure* λ_k in $\mathbb{R}^k$ is *normal / regular?*, and
  $\forall A \in \mathcal{M}_{\lambda_k^*} : \lambda(A) = \sup\{\lambda(K) \mid K \underset{compact}{\subset} X, K \subset A\}$ ⟨thm. 4.2 p. 37 ∈ [1]⟩

- It can be proved that the *Lebesgue measure* in $(\mathbb{R}^k, \mathcal{M}_{\lambda^*})$ is
  the *completion* of the *Lebesgue measure* in $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$
  by showing that $\mathcal{M}_{\lambda_k^*} = \mathcal{B}(\mathbb{R}^k)_\lambda$
  using *thm. 2.10* and the *normality* of λ. ⟨rem. 4.3 p. 38 ∈ [1]⟩

- For *metric space* X: *Every measure* in the *metric space* $(X, \mathcal{B}(X))$ is called ***Borel measure* (in X)** ⟨notes from lessons ∈ []⟩

- The *Lebesgue measure* in $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ is
  the *unique Borel measure* in $\mathbb{R}^k$ such that $\forall I \underset{interval}{\subset} \mathbb{R}^k : \lambda(I) = v(I)$.
  ⟨thm. 4.4 p. 39 ∈ [1]⟩

- The *exterior Lebesgue measure* and the *Lebesgue measure* are
  motion invariant (see section 4.1 for notation and basic stuff):

  - $\forall A \subset \mathbb{R}^k, x \in \mathbb{R}^k : \lambda_k^*(A) = \lambda_k^*(A + x)$. ⟨thm 4.6 (i) p. 39 ∈ [1]⟩
  - $\forall A \subset \mathbb{R}^k, x \in \mathbb{R}^k : A \in \mathcal{M}_{\lambda_k^*} \Leftrightarrow A + x \in \mathcal{M}_{\lambda_k^*}$. ⟨thm 4.6 (ii α) p. 39 ∈ [1]⟩
    $\forall A \in \mathcal{M}_{\lambda_k^*} : \lambda_k(A) = \lambda_k(A + x)$. ⟨thm 4.6 (ii β) p. 39 ∈ [1]⟩
  - Also holds for the *Lebesgue measure* in $\mathcal{B}(\mathbb{R}^k)$: ⟨rem. 4.7 p. 40 ∈ [1]⟩
    $\forall A \subset \mathbb{R}^k, x \in \mathbb{R}^k : A \in \mathcal{B}(\mathbb{R}^k) \Leftrightarrow A + x \in \mathcal{B}(\mathbb{R}^k)$.
    $\forall A \in \mathcal{B}(\mathbb{R}^k) : \lambda_k(A) = \lambda_k(A + x)$.

- *Uniqueness* of the *Lebesque measure* w.r.t. *motion invariance*:
  If μ is a *Borel measure* in $\mathbb{R}^k$ such that
  $\forall I \underset{interval}{\subset} \mathbb{R}^k : \forall x \in \mathbb{R}^k : \mu(I + x) = \mu(I)$ and such that
  $\forall K \underset{compact}{\subset} \mathbb{R}^k : \mu(K) < \infty$, then
  $\exists \alpha \in \mathbb{R}, \alpha \geq 0 : \mu = \alpha \cdot \lambda$, i.e.:
  $\forall A \in \mathcal{B}(\mathbb{R}^k) : \mu(A) = \alpha \cdot \lambda(A)$. ⟨thm. 4.9 p. 41 ∈ [1]⟩

- $\forall A \in \mathcal{M}_{\lambda_k^*}, \lambda(A) > 0 : \exists \delta > 0 : S(0, \delta) \subset A - A$.
  (*Steinhaus*) ⟨thm. 4.10 p. 41 ∈ [1]⟩

- The *Cantor set* is *over countable* (so it has *cadinality* of the *continuum*) and has *Lebesgue measure* 0. ⟨thm. 4.11 p. 42 ∈ [1]⟩

- Vitali's theorem: There *exists* a *subset* of $\mathbb{R}$, and hence also of $]0, 1[$,
  which is *not Lebesgue measurable.* (hence: $\mathcal{M}_{\lambda^*} \subsetneq \mathcal{P}(\mathbb{R})$).
  Also holds for $\mathbb{R}^k$. ⟨thm. 4.12 p. 42 ∈ [1]⟩

- There *exists* a *Lebesgue measurable subset* of $\mathbb{R}$ which is *not a Borel set.* (hence: $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{M}_{\lambda^*}(\mathbb{R})$). Also holds for $\mathbb{R}^k$. ⟨thm. 4.14 p. 43 ∈ [1]⟩

- For (X, ρ) *metric space* and μ *limited Borel measure* in X: $\forall A \in \mathcal{B}(X)_\mu : \overline{\mu}(A) =$ $\inf\{\mu(G) \mid G \underset{open}{\subset} X, A \subset G\} =$ $\sup\{\mu(F) \mid F \underset{closed}{\subset} X, F \subset A\}$ ⟨thm. 4.15 p. 44 ∈ [1]⟩

- A *completely limited metric space* $(X, \rho) \overset{def}{=}$ A *metric space* (X, ρ) where $\forall \epsilon > 0 : \exists \{x_1, \dots, x_n\} \subset X : X = \bigcup_{n=1}^{\infty} S(x_i, \epsilon)$. Equivalently: X is a *finite union* of sets with *diameter* $< \epsilon$. ⟨p. 45 ∈ [1]⟩

- Any *complete* and *completely limited metric space* is *compact.* ⟨lemma 4.16 p. 45 ∈ [1]⟩

- A *Polish metric space* $(X, \rho) \overset{def}{=}$ A *metric space* which is *separable* and which has an *equivalent complete metric.* ⟨p. 45 ∈ [1]⟩

- For *Polish metric space* X: Every *limited Borel measure* μ is *normal / regular?.* ⟨thm. 4.17 p. 45 ∈ [1]⟩

3.2 From Excercises

- For an $A \subset \mathbb{R}^k$, the *following are equivalent*: ⟨5 ∈ [1]⟩

  1. $A \in \mathcal{M}_{\lambda^*}$
  2. $A = B \cup C$ for *some* F_σ*-set* B and *some set* C with $\mu(C) = 0$
  3. There *exists* an F_σ*-set* D, such that $\lambda^*(A \triangle D) = 0$

- For $A \subset \mathbb{R}, \alpha \in \mathbb{R}, \delta > 0$: If $\forall t \in \mathbb{R}$ with $|t| < \delta : \alpha + t \in A$ or $\alpha - t \in A$, then $\lambda^*(A) \geq \delta$ ⟨6 ∈ [1]⟩

- If $A \subset \mathbb{R}$ and $\alpha \in]0, 1[$ such that $\forall I \underset{interval}{\subset} \mathbb{R} : \lambda^*(A \cap I) \leq \alpha\lambda(I)$, then $\lambda^*(A) = 0$ ⟨7 ∈ [1]⟩

3.3 Measurable Functions

- For *measurable space* $(X, \mathcal{A})$: $f : X \to [-\infty, \infty]$ **is $\mathcal{A}$-measurable** $\stackrel{def}{=}$ $\forall \beta \in \mathbb{R}$: the set $[f \leq \beta] \equiv \{x \in X \mid f(x) \leq \beta\} \equiv f^{-1}([\infty, \beta])$ is *measurable* (i.e. it *belongs to* $\mathcal{A}$). ⟨def. 5.1 p. 51 ∈ [1]⟩

- For *measurable space* $(X, \mathcal{A})$ and μ is a *measure* in $(X, \mathcal{A})$:
  $f : X \to [-\infty, \infty]$ **is μ-measurable** $\stackrel{def}{=} f$ is $\mathcal{A}_\mu$-*measurable.*
  ⟨def. 5.1 p. 51 ∈ [1]⟩

- For *measurable space* $(\mathbb{R}^k, \mathcal{A})$ and λ is the *Lebesgue measure* in $(\mathbb{R}^k, \mathcal{A})$:
  $f : \mathbb{R}^k \to [-\infty, \infty]$ **is *Lebesgue measurable*** $\stackrel{def}{=} f$ is λ-*measurable.*
  ⟨def. 5.1 p. 51 ∈ [1]⟩

- For *metric space* X:
  $f : X \to [-\infty, \infty]$ **is *Borel measurable*** $\stackrel{def}{=} f$ is $\mathcal{B}(X)$-*measurable.*
  ⟨def. 5.1 p. 51 ∈ [1]⟩

- For *mesurable space* $(X, \mathcal{A})$ and $f : X \to [-\infty, \infty]$,
  the *following are equivalent*:

  1. f is $\mathcal{A}$-*measurable*
  2. $\forall \beta \in \mathbb{R} : [f \leq \beta] \equiv f^{-1}([-\infty, \beta]) \in \mathcal{A}$ (def. of *measurable*)
  3. $\forall \beta \in \mathbb{R} : [f < \beta] \equiv f^{-1}([-\infty, \beta[) \in \mathcal{A}$
  4. $\forall \beta \in \mathbb{R} : [f \geq \beta] \equiv f^{-1}([\beta, \infty]) \in \mathcal{A}$
  5. $\forall \beta \in \mathbb{R} : [f > \beta] \equiv f^{-1}(]\beta, \infty]) \in \mathcal{A}$

  ⟨thm. 5.2 p. 51 ∈ [1]⟩

- For *measurable space* $(X, \mathcal{A})$ and $B \subset X$:
  $B \in \mathcal{A} \Leftrightarrow \chi_B$ is $\mathcal{A}$-*measurable.* ⟨rem. 5.3 (i) p. 52 ∈ [1]⟩
  (χ_B is the *indicator function* of B, see section 4).
  Proof: $[\chi_B \leq B] = \begin{cases} \emptyset & , \beta < 0 \\ X \backslash B & , 0 \leq \beta < 0 \\ X & , 1 \leq B \end{cases}$

- Any *Borel-measurable* function $f : \mathbb{R}^k \to \mathbb{R}$ is also *Lebesgue measurable.*
  Proof: $\mathcal{B}(\mathbb{R}^k) \subset \mathcal{M}_{\lambda_k^*}$. ⟨rem. 5.3 (ii) p. 52 ∈ [1]⟩
  Warning: The *opposite* does *not hold.* E.g.: $A \in \mathcal{M}_{\lambda_k^*} \backslash \mathcal{B}(\mathbb{R}^k)$ and $f = \chi_A$.

- Any *continuous function* $f : \mathbb{R}^k \to \mathbb{R}$ is *Borel-measurable.*
  Proof: $\forall \beta \in \mathbb{R} : [f \leq \beta] \underset{closed}{\subset} \mathbb{R}^k$, so $[f \leq \beta]$ is *Borel.* ⟨rem. 5.3 (iii) p. 52 ∈ [1]⟩

- For $I \underset{interval}{\subset} \mathbb{R}$ and $f : I \to \mathbb{R}$ *increasing*: f is *Borel measurable.*
  Proof: $\forall \beta \in \mathbb{R}$: set $\alpha = \sup[f \leq \beta]$ and then
  $[f \leq \beta] = [-\infty, \alpha[\cap I$ or $[f \leq \beta] = [-\infty, \alpha] \cap I$, so $[f \leq \beta]$ is *Borel.*
  ⟨rem. 5.3 (iv) p. 51 ∈ [1]⟩

- For *measurable space* $(X, \mathcal{A})$ and $C \subset X$:
  - **The *trace* $\mathcal{A}_C$ of $\mathcal{A}$ in C** $\stackrel{def}{=}$ the family $\mathcal{A}_C = \{A \cap C \mid A \in \mathcal{A}\}$. ⟨def. 5.4 p. 52 ∈ [1]⟩
  - $\mathcal{A}_C$ is a *σ-algebra* in C. ⟨def. 5.4 p. 52 ∈ [1]⟩
  - $f : C \to [-\infty, \infty]$ **is *measurable*** $\stackrel{def}{=}$ f is $\mathcal{A}_C$*-measurable* (i.e.: $\forall \beta \in \mathbb{R} : [f \leq \beta] \in \mathcal{A}_C$). ⟨def. 5.4 p. 52 ∈ [1]⟩
  - If $C \in \mathcal{A}$ then $\mathcal{A}_C = \{B \in \mathcal{A} \mid B \subset C\}$, so f is *measurable* $\Leftrightarrow \forall \beta \in \mathbb{R} : [f \leq \beta] \in \mathcal{A}$. ⟨def. 5.4 p. 52 ∈ [1]⟩
- For $(X, \mathcal{A})$ *measurable space* and $f : X \to [-\infty, \infty]$:
  - f *measurable* and $C \subset X \Rightarrow f|_C$ *measurable*. ⟨thm. 5.5 (i) p. 53 ∈ [1]⟩
  - For $\{C_n\}$ *sequence* in $\mathcal{A}$ and $\cup_{n=1}^{\infty} C_n = X$: f *measurable* $\Leftrightarrow \forall n \in \mathbb{N} : f|_{C_n}$ *measurable*. ⟨thm. 5.5 (ii) p. 53 ∈ [1]⟩
- For *metric space* X and $Y \underset{subspace}{\subset} X$:
  - $\mathcal{B}(X)_Y = \mathcal{B}(Y)$. ⟨thm. 5.6 (i) p. 53 ∈ [1]⟩
    I.e.: The trace of $\mathcal{B}(X)$ in Y equals the σ-algebra of the Borel set $\mathcal{B}(Y)$.
  - $f : X \to [-\infty, \infty]$ is *Borel measurable* $\Rightarrow$ $f|_Y : Y \to [-\infty, \infty]$ *Borel measurable*. ⟨thm. 5.6 (ii) p. 53 ∈ [1]⟩
- For *measurable space* $(X, \mathcal{A})$:
  - For function $f : X \to \mathbb{R}$ the *following are equivalent*: ⟨thm. 5.7 p. 53 ∈ [1]⟩
    1. f is *measurable*.
    2. $\forall G \underset{open}{\subset} \mathbb{R} : [f \in G] = f^{-1}(G) \in \mathcal{A}$.
    3. $\forall F \underset{closed}{\subset} \mathbb{R} : [f \in F] = f^{-1}(F) \in \mathcal{A}$.
    4. $\forall B \in \mathcal{B}(X) : [f \in B] = f^{-1}(B) \in \mathcal{A}$.
  - For *measurable functions* $f, g : X \to [-\infty, \infty]$ the following holds:
    1. $[f < g] = \{x \in X \mid f(x) < g(x)\} \in \mathcal{A}$. ⟨thm. 5.8 p. 54 ∈ [1]⟩
    2. $[f \leq g] = \{x \in X \mid f(x) \leq g(x)\} \in \mathcal{A}$. ⟨thm. 5.8 p. 54 ∈ [1]⟩
    3. $[f = g] = \{x \in X \mid f(x) = g(x)\} \in \mathcal{A}$. ⟨thm. 5.8 p. 54 ∈ [1]⟩
    4. $f \wedge g$ is *measurable* where $\forall x \in X : (f \wedge g)(x) = \min\{f(x), g(x)\}$. ⟨thm. 5.9 (i) p. 54 ∈ [1]⟩
    5. $f \vee g$ is *measurable* where $\forall x \in X : (f \vee g)(x) = \max\{f(x), g(x)\}$. ⟨thm. 5.9 (i) p. 54 ∈ [1]⟩
    6. $f^+ = f \vee 0$ is *measurable*. ⟨thm. 5.9 (ii) p. 54 ∈ [1]⟩
    7. $f^- = (-f) \vee 0$ is *measurable*. ⟨thm. 5.9 (ii) p. 54 ∈ [1]⟩
  - For $f_n : X \to [-\infty, \infty]$ and $\{f_n\}_{n \in \mathbb{N}}$ a *sequence* of *measurable* functions:
    1. $\sup_n f_n$ is *measurable*. ⟨thm. 5.10 (i) p. 55 ∈ [1]⟩
    2. $\inf_n f_n$ is *measurable*. ⟨thm. 5.10 (i) p. 55 ∈ [1]⟩
    3. $\limsup_n f_n$ is *measurable*. ⟨thm. 5.10 (ii) p. 55 ∈ [1]⟩
    4. $\liminf_n f_n$ is *measurable*. ⟨thm. 5.10 (ii) p. 55 ∈ [1]⟩
    5. $\lim_n f_n$ is *measurable if it exists*. ⟨thm. 5.10 (iii) p. 55 ∈ [1]⟩
       I.e.: If $\lim_{n \to \infty} f_n$ *converges* to a *point* $f = \limsup_n f_n = \liminf_n f_n$.

- For *measurable space* $(X, \mathcal{A})$:
  - If X is a *metric space* and $f : X \to \mathbb{R}$:
    1. f **is *Baire-1*** $\overset{def}{=}$ ⟨def. 5.11 p. 55 ∈ [1]⟩
       It is *point of convergence* of a *sequence* of *continuous functions.*
    2. f **is *Baire-2*** $\overset{def}{=}$ ⟨def. 5.11 p. 55 ∈ [1]⟩
       It is *point of convergence* of a *sequence* of *Baire-1 functions.*
    3. *Baire-1* and *Baire-2* functions are *measurable.*
       Proof: Thm. 5.3 (iii) and thm. 5.10.
    4. It is known from real analysis that:
       $f = \chi_{\mathbb{Q}}$ is *Baire-2* but *not Baire-1.*
    5. The *first derivative* f' of a *differentiable function* f is *Baire-1.*
       ⟨def. 5.11 p. 55, exc. 5-10 ∈ [1]⟩
  - For *measurable functions* $f, g : X \to [0, \infty]$ and $\alpha \in \mathbb{R}, \alpha \geq 0$
    the following holds:
    1. $\alpha \cdot f$ is *measurable.* ⟨thm. 5.12 (i) p. 55 ∈ [1]⟩
    2. $f + g$ is *measurable.* ⟨thm. 5.12 (ii) p. 55 ∈ [1]⟩
  - For *measurable functions* $f, g : X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$
    the following holds:
    1. $\alpha \cdot f$ is *measurable.* ⟨thm. 5.13 (i) p. 56 ∈ [1]⟩
    2. $f + g$ and $f - g$ are *measurable.* ⟨thm. 5.13 (ii) p. 56 ∈ [1]⟩
    3. $f \cdot g$ is *measurable.* ⟨thm. 5.13 (iii) p. 56 ∈ [1]⟩
    4. $h(x) = \begin{cases} \frac{f(x)}{g(x)} & , g(x) \neq 0 \\ 0 & , g(x) = 0 \end{cases}$ is *measurable.* ⟨thm. 5.13 (iv) p. 56 ∈ [1]⟩
    5. $|f|$ is *measurable.* ⟨thm. 5.13 (v) p. 56 ∈ [1]⟩
  - $s : X \to \mathbb{R}$ **is *simple*** $\overset{def}{=}$
    The set $s(X)$ of values of s is *finite.* ⟨def. 5.14 p. 57 ∈ [1]⟩
  - For $s : X \to \mathbb{R}$ *simple*: ⟨def. 5.14 p. 57 ∈ [1]⟩
    There *exists* a *unique finite measurable partition* $\{A_1, \dots, A_n\}$ of X
    with $A_i \neq \emptyset$ (i.e.: A_i *pairwise disjoint* and $X = \bigcup A_i$) and
    unique $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $\forall i \neq j : \alpha_i \neq \alpha_j$ such that:
    $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$. **This is called the *normal form* of** s.
    It is found by setting $\{\alpha_1, \dots, \alpha_n\} = s(X)$ where $\forall i \neq j : \alpha_i \neq \alpha_j$
    and setting $\forall i \in \{1, \dots, n\} : A_i = s^{-1}(\{\alpha_i\})$
  - For *measurable function* $f : X \to [0, +\infty]$:
    There *exists* an *increasing sequence* of functions $\{s_n\}$
    (i.e.: $\forall x \in X : \forall n \in \mathbb{N} : s_n(x) \leq s_{n+1}(x)$) such that
    $f = \lim_{n\to\infty} s_n$ and $\forall n \in \mathbb{N} : s_n : X \to [0, \infty[$ is *simple.*
    If f is *limited*, then $\lim_{n\to\infty} s_n$ *converges uniformly.* ⟨thm. 5.15 p. 57 ∈ [1]⟩
  - For *measurable function* $f : X \to [-\infty, +\infty]$:
    There *exists* a *sequence* of *simple* functions $\{s_n\}$ such that
    $f = \lim_{n\to\infty} s_n$ and $\{|s_n|\}$ is *increasing.*
    If f is *limited*, then $\lim_{n\to\infty} s_n$ *converges uniformly.* ⟨thm. 5.16 p. 58 ∈ [1]⟩

- For *measurable space* $(X, \mathcal{A})$:
  - $f : X \to \mathbb{C}$ **is $\mathcal{A}$-measurable** a.k.a. **measrurable (w.r.t. $\mathcal{A}$)** $\stackrel{def}{=}$ $\forall B \in \mathcal{B}(\mathbb{C}) : f^{-1}(B) \in \mathcal{A}$. ⟨def. 5.17 p. 59 ∈ [1]⟩
  - For a *measure* μ in $(X, \mathcal{A})$:
    $f : X \to \mathbb{C}$ **is μ-measurable** $\stackrel{def}{=}$
    f is $\mathcal{A}_\mu$-*measurable*. ⟨def. 5.17 p. 59 ∈ [1]⟩
  - $f : X \to \mathbb{C}$ **is *Borel measurable*** $\stackrel{def}{=}$
    f is $\mathcal{B}(X)$-*measurable*. ⟨def. 5.17 p. 59 ∈ [1]⟩
  - For $f : X \to \mathbb{C}$, the *following are equivalent*: ⟨p. 59 ∈ [1]⟩
    1. f is $\mathcal{A}$-*measurable*
    2. $\forall G \underset{open}{\subset} \mathbb{R} : f^{-1}(G) \in \mathcal{A}$.
    3. $\forall F \underset{closed}{\subset} \mathbb{R} : f^{-1}(F) \in \mathcal{A}$.
  - For $f : X \to \mathbb{C}$, $f = u + iv$ where
    $u, v : X \to \mathbb{R}$ are the *real* and *imaginary* parts of f:
    1. f *measurable* $\Leftrightarrow$ u, v *measurable*. ⟨thm. 5.18 (i) p. 59 ∈ [1]⟩
       Equivalently: u^+, u^-, v^+ and v^- are *measurable*.
    2. f *measurable* $\Rightarrow$ there *exists* $\alpha : X \to \mathbb{C}$ *measurable* such that:
       $\forall x \in X : |\alpha(x)| = 1$ and $f = \alpha \cdot |f|$. ⟨thm. 5.18 (ii) p. 59 ∈ [1]⟩
  - For a *sequence* $\{f_n\}_{n\in\mathbb{N}}$ of *measurable* functions $f_n : X \to \mathbb{C}$:
    $\lim_{n\to\infty} f_n = f \Rightarrow f$ *measurable*. ⟨thm. 5.19 p. 60 ∈ [1]⟩
  - For $f, g : X \to \mathbb{C}$ *measurable* and $c \in \mathbb{C}$ we have: ⟨thm. 5.20 p. 60 ∈ [1]⟩
    1. $c \cdot f$, $f + g$, $f \cdot g$ and $|f|$ are *measurable*.
    2. $\frac{f}{g}$ is *measurable* if $\forall x \in X : g(x) \neq 0$.
  - A *measurable* function $s : X \to \mathbb{C}$ **is *simple*** $\stackrel{def}{=}$
    $s(X)$ is *finite*. ⟨def. 5.21 p. 60 ∈ [1]⟩
  - $s : X \to \mathbb{C}$ is *simple* $\Leftrightarrow$ $Re\ f$ and $Im\ f$ are *simple*. ⟨def. 5.21 p. 60 ∈ [1]⟩
  - For $f : X \to \mathbb{C}$ *measurable*:
    There *exists* a *sequence* $\{s_n\}_{n\in\mathbb{N}}$ of *simple* functions
    $s_n : X \to \mathbb{C}$ such that $f = \lim_{n\to\infty} s_n$.
    And: f *limited* $\Rightarrow$ $\{s_n\}$ *converges uniformly*. ⟨thm. 5.22 p. 61 ∈ [1]⟩

- For *measure space* $(X, \mathcal{A}, \mu)$:
  - For *simple function* $f : X \to [0, \infty[$ with *normal form* $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$:
    The *Lebesgue integral* $\int f d\mu$ of f (w.r.t. μ) $\stackrel{def}{=} \sum_{i=1}^{n} \alpha_i \mu(A_i)$.
    If the limit $0 \cdot (+\infty)$ appears we define $0 \cdot (+\infty) = 0$.
    Clearly $\int f d\mu \in [0, \infty]$. ⟨def. 6.1 p. 65 ∈ [1]⟩
  - For *simple function* $f : X \to [0, \infty[$ and
    $B_1, \ldots, B_n \in \mathcal{A}$ *pairwise disjoint* such that:
    $f = \sum_{j=1}^{n} B_j \chi_{B_j}$, then:
    $\sum_{j=1}^{n} B_j \mu(B_j)$. ⟨lemma. 6.2 p. 65 ∈ [1]⟩
  - For $f, g \geq 0$ *simple functions* and $\alpha \geq 0$:
    * $\int \alpha f d\mu = \alpha \int f d\mu$. ⟨thm. 6.3 (i) p. 66 ∈ [1]⟩
    * $\int (f + g) d\mu = \int f d\mu + \int g d\mu$. ⟨thm. 6.3 (ii) p. 66 ∈ [1]⟩
    * $f \leq g \Rightarrow \int f d\mu \leq \int g d\mu$. ⟨thm. 6.3 (iii) p. 66 ∈ [1]⟩

- For *measurable function* $f : X \to [0, \infty]$: ⟨def. 6.4 p. 67 ∈ [1]⟩
  The *Lebesgue integral* $\int f d\mu$ of f (w.r.t. μ) $\stackrel{def}{=}$
  $\int f d\mu = \sup\{\int s d\mu \mid s \text{ simple}, 0 \leq s \leq f\}$.
  If $A \in \mathcal{A}$, **the *Lebesgue integral* $\int_A f d\mu$ of f (w.r.t. μ)** $\stackrel{def}{=}$
  $\int_A f d\mu = \int f \chi_A d\mu$. Clearly: $\int_A f d\mu \in [0, \infty]$.

- $\int f d\mu = \int_X f d\mu$. ⟨rem. 6.5 p. 67 ∈ [1]⟩

- $\forall A \in \mathcal{A} : \int_A f d\mu = \int f|_A d\mu|_A$, where
  $\mu|_A$ is μ *restricted* to the *trace* $\mathcal{A}_A$ of $\mathcal{A}$ in A. ⟨rem. 6.5 p. 67 ∈ [1]⟩

- For $f, g : X \to [0, \infty]$ *measurable* functions, $\alpha \geq 0$ and $A, B \in \mathcal{A}$:
  - $\int \alpha f d\mu = \alpha \int f d\mu$. ⟨thm. 6.6 (i) p. 67 ∈ [1]⟩
  - $f \leq g \Rightarrow \int f d\mu \leq \int g d\mu$. ⟨thm. 6.6 (ii) p. 67 ∈ [1]⟩
  - $A \subseteq B \Rightarrow \int_A f d\mu \leq \int_B f d\mu$. ⟨thm. 6.6 (iii) p. 67 ∈ [1]⟩
  - If $\mu(A) = 0$ or $f = 0$ in A, then $\int_A f d\mu = 0$. ⟨thm. 6.6 (iv) p. 67 ∈ [1]⟩

- For *simple function* $s : X \to [0, \infty[$: ⟨lemma 6.7. p. 68 ∈ [1]⟩
  If we set $v : \mathcal{A} \to [0, \infty]$ to $v(A) = \int_A s d\mu$, then v is a *measure*.

- *Monotonic Convergence of Lebesgue*: ⟨thm. 6.8 p. 69 ∈ [1]⟩
  For $f_n : X \to [0, \infty]$ *increasing sequence* of *measurable* functions
  (i.e. $f_1 \leq f_2 \leq \cdots$):
  If we set $f = \lim_{n\to\infty} f_n$, then
  $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$.

3.4 More...

- Theorem 15 is generalized as follows:
  If $f, g \in \mathcal{L}'_{\mathbb{R}}(\mu)$ and $f \leq g$ μ-almost everywhere, then
  $\int f d\mu \leq \int g d\mu$ ⟨rem. 1 ∈ [1]⟩

- In the teorem K.Σ., the requirement? that $|f_n| \leq g$ can be replaced by $|f_n| \leq$ μ-almost everywhere ⟨rem. 2 ∈ [1]⟩

- Also the requirement? that $\{f_n\}$ *converges* towards f can be replaced by ? > ? μ-almost everywhere ⟨rem. 2 ∈ [1]⟩

- For *measure space* $(X, \mathcal{A}, \mu)$ and $f_n : X \to [0, \infty]$ for $n = 1, 2, \ldots$ is a *decreasing sequence* of *measurable functions* where $\int f_1 d\mu < \infty$, then:
  $\int \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int f_n d\mu < \infty$. ⟨exc. 6 ∈ [1]⟩

- Generalization of Fatou's theorem: For *measure space* $(X, \mathcal{A}, \mu)$ and $h : X \to [0, \infty]$ *measurable* with $\int h d\mu < \infty$ and $f_n : X \to [-\infty, \infty]$ for $n = 1, 2, \ldots$ is a *sequence* of *measurable functions* where $\forall n \in \mathbb{N} : f_n \geq -h$:
  $\int \lim_{n\to\infty} \inf f_n d\mu \leq \lim_{n\to\infty} \inf \int f_n d\mu$ ⟨exc. 7 ∈ [1]⟩

3.5 Convergence of the Riemann Integral towards the Lebesgue Integral

- $f : [\alpha, \beta] \to \mathbb{R}$ **is *Riemann integrable*** $\stackrel{def}{=}$
  the *limits* of the *upper sum* and *lower sum* are *equal*, written:
  $\int_\alpha^\beta f$ or $\int_\alpha^\beta f(x)dx$

- Riemann criterion: $f : [\alpha, \beta] \to \mathbb{R}$ is *Riemann integrable* $\Leftrightarrow$
  $\forall \epsilon > 0 : \exists$ *partition* ρ of $[\alpha, \beta] : U(f, \rho) - L(f, \rho) < \epsilon$

- From real analysis about Riemann integrability:
  For *metric spaces* X, Y, $f : X \to Y$, $x \in X$:
  f *continuous* at $x \Leftrightarrow$
  $\mathcal{T}_{f(x)} \equiv \inf\{\textit{partition } f(U) \mid U \textit{ neighborhood of } x\} = 0$

- For $f : [\alpha, \beta] \to \mathbb{R}$ *discontinuous* we have:
  - f is *Riemann integrable* $\Leftrightarrow$
    the set $A(f) = \{x \in [\alpha, \beta] \mid f$ is *not continuous* at $x\}$ has *Lebesgue measure* 0. (i.e.: f is λ-almost everywhere continuous)
  - If f is *Riemann integrable*, then
    f is *Lebesgue integrable* (w.r.t. λ) and $\int_\alpha^\beta f = \int_{[\alpha,\beta]} f d\lambda$
  - $A(f) = [\mathcal{T}_f > 0] = \cup_{n=1}^\infty [\mathcal{T}_f \geq \frac{1}{n}] = f_\sigma$ *closed*

3.6 Results from Excercises

- For *measurable space* $(X, \mathcal{A})$: ⟨exc. 5 ∈ [1]⟩
  If $\{\mu_n\}$ is a *growing sequence* of *measure functions* in $(X, \mathcal{A})$
  (i.e.: $\forall A \in \mathcal{A} : \forall n \in \mathbb{N} : \mu_n(A) \leq \mu_{n+1}(A)$), then
  $\forall A \in \mathcal{A} : \mu(A) = \lim_n \mu_n(A) : \mathcal{A} \to [0, \infty]$ is a *measure*.

4 Misc Notation

- For a *sequence* of *subsets* $\{A_n\}$ of a *set* X: ⟨p. 3 ∈ [1]⟩
  - A_n ***growing / increasing sequence*** $\stackrel{def}{=} \forall n : A_n \subset A_{n+1}$
  - A_n ***falling / decreasing sequence*** $\stackrel{def}{=} \forall n : A_n \supset A_{n+1}$
- *Indicator function* of $\mathcal{B}$: $\mathcal{X}_{\mathcal{B}}(x) = \begin{cases} 1 & \text{, for } x \in \mathcal{B} \\ 0 & \text{, for } x \notin \mathcal{B} \end{cases}$
- For *partition* $\Delta = \{\alpha = t_0 < t_1 < \cdots < t_n = \beta\}$ of $[\alpha, \beta]$: the *granularity* of $\Delta \stackrel{def}{=} ||\Delta|| = \max_{i \in \{1,\dots,n\}} t_i - t_{i-1}$
- For *partitions* Δ_1, Δ_2 of $[\alpha, \beta]$: Δ_2 **is *finer* than** $\Delta_1 \stackrel{def}{=} \Delta_1 \subset \Delta_2$ (also written $\Delta_1 \prec \Delta_2$)

4.1 Adding And Subtracting Sets, Motions etc.

- For $A, B \subset \mathbb{R}^k$, $x \in \mathbb{R}^k$ we define: ⟨def. 4.5 p. 39 ∈ [1]⟩
  - $A \pm B = \{x \pm y \mid x \in A, y \in B\}$
  - $A \pm x = A \pm \{x\}$
- For *any* $\{A_n\}$ *sequence* of *subsets* of $\mathbb{R}^k$ and $A, B \subset \mathbb{R}^k$ and $x \in \mathbb{R}^k$: ⟨p. 39 ∈ [1]⟩
  - $(\cup_{n=1}^{\infty} A_n) \pm x = \cup_{n=1}^{\infty} (A_n \pm x)$
  - $(\cap_{n=1}^{\infty} A_n) \pm x = \cap_{n=1}^{\infty} (A_n \pm x)$
  - $(A \backslash B) \pm x = (A \pm x) \backslash (B \pm x)$
- ***Motion with*** x: $T_x : \mathbb{R}^k \to \mathbb{R}^k$ with $\forall y \in \mathbb{R}^k : T_{\pm x}(y) = y \pm x$. $T_x(A) = A + x$ and $T_{-x}(A) = A - x$. ⟨def. 4.5 p. 39 ∈ [1]⟩
  - T_x is a *homeomorphism*, i.e.: $\forall A \subset \mathbb{R}^k : A$ *open* $\Leftrightarrow A + x$ *open*.
  - T_x is *bijective* and using previous identities we also get: $\forall A \subset \mathbb{R}^k : A \in \mathbb{B}(\mathbb{R}^k) \Leftrightarrow A + x \in \mathbb{B}(\mathbb{R}^k)$

References

[1] Γ. Κουμουλλής, Σ. Νεγρεπόντης. *Θεωρία Μέτρου*, Εκδόσεις Συμμετρία Αθήνα 1991.