Maths Foundation Summary

Ánoq of the Sun, Hardcore Processing *

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1 Functions

Premises: A, B ***sets***

- For $f : A \to B$; $g : B \to A$ **is a *left inverse* for** $f \stackrel{def}{=} g \circ f = id_A$. The left inverse is *unique.* ⟨exc. 1.2,5 p. 21 ∈ [1]⟩
- For $f : A \to B$; $g : B \to A$ **is a *right inverse* for** $f \stackrel{def}{=} f \circ g = id_A$. The right inverse is *not necessarily unique.* ⟨exc. 1.2,5 p. 21 ∈ [1]⟩
- $f : A \to B$ ***injective*** $\stackrel{def}{=} f(a) = f(a') \Rightarrow a = a'$ ⟨? ∈ [1]⟩
- $f : A \to B$ ***surjective*** $\stackrel{def}{=} b \in B \Rightarrow \exists a \in A : b = f(a)$ ⟨? ∈ [1]⟩
- $\exists f : A \to B$ *injective* $\Leftrightarrow \exists g : B \to A$ *surjective* ⟨p. 50, proof thm. 7.8 ∈ [1]⟩
- f, g *injective* $\Rightarrow f \circ g$ *injective* ⟨exc. 1.2,4 p. 21 ∈ [1]⟩
- f, g *surjective* $\Rightarrow f \circ g$ *surjective* ⟨exc. 1.2,4 p. 21 ∈ [1]⟩

1.1 Identity, Projection and Coordinate Functions

Premises: A, B ***sets***

- **The *identity function*** id_A **for a *set*** $A \stackrel{def}{=} \forall a \in A : id_A(a) = a$
- ***Projection functions*** / ***projection mapping*** $\stackrel{def}{=}$
  $\pi_i : X_1 \times X_2 \times \cdots \times X_i \times \cdots \times X_n \to X_i$, defined by:
  $\pi_i(x_1, x_2, \ldots, x_i, \ldots, x_n) = x_i$. *Surjective* if all $X_i \neq \emptyset$. ⟨p. 87 ∈ [1]⟩
  Arbitrary index: $\pi_\beta : \prod_{\alpha \in J} A_\alpha \to A_\beta$, where $\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$ ⟨p. 144 ∈ [1]⟩
- ***Coordinate functions*** **of** $f \stackrel{def}{=}$
  $f_1, f_2, \ldots, f_n$, where $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ ⟨p. 110 ∈ [1]⟩

1.2 Images and Preimages through Functions

Premises: A, B ***sets***, $A_0, A_1 \subset A, B_0, B_1 \subset B, f : A \to B$

- **The *image* $f(A_0)$ through a function** $f \overset{def}{=} \{f(x)|x \in A_0\}$. I.e.: $y \in f(A_0) \Leftrightarrow \exists x \in A_0 : y = f(x)$. ⟨formula 1 ∈ [2]⟩

- **The *preimage* $f^{-1}(B_0)$ through** $f \overset{def}{=} \{x \in A|f(x) \in B_0\}$. I.e.: $x \in f^{-1}(B_0) \Leftrightarrow f(x) \in B_0$. ⟨formula 2 ∈ [2]⟩

- The *preimage* through a function preserves: $\subset, \cup, \cap, \setminus$. I.e.:
  $$\begin{array}{lcl} B_0 \subset B_1 & \Rightarrow & f^{-1}(B_0) \subset f^{-1}(B_1) \\ f^{-1}(B_0 \cup B_1) & = & f^{-1}(B_0) \cup f^{-1}(B_1) \\ f^{-1}(B_0 \cap B_1) & = & f^{-1}(B_0) \cap f^{-1}(B_1) \\ f^{-1}(B_0 \setminus B_1) & = & f^{-1}(B_0) \setminus f^{-1}(B_1) \end{array}$$
  ⟨exc. 1.2, 27 ∈ [1]⟩

- The *image* through a function preserves only: $\subset, \cup$. I.e.:
  $$\begin{array}{lcll} A_0 \subset A_1 & \Rightarrow & f(A_0) \subset f(A_1) & \\ f(A_0 \cup A_1) & = & f(A_0) \cup f(A_1) & \\ f(A_0 \cap A_1) & \subset & f(A_0) \cap f(A_1) & \textit{equals} \text{ if } f \textit{ injective} \\ f(A_0 \setminus A_1) & \supset & f(A_0) \setminus f(A_1) & \textit{equals} \text{ if } f \textit{ surjective} \end{array}$$
  ⟨exc. 1.2, 27 ∈ [1]⟩

- f *injective* $\Rightarrow A_0 = f^{-1}(f(A_0))$ ⟨p. 19 ∈ [1]⟩
  $A_0 \subset f^{-1}(f(A_0))$ *always holds.*

- f *surjective* $\Rightarrow B_0 = f(f^{-1}(B_0))$ ⟨p. 19 ∈ [1]⟩
  $B_0 \supset f(f^{-1}(B_0))$ *always holds.*

- If there *exists functions* $g : B \to A$ and $h : B \to A$ such that:
  $\forall a \in A : g(f(a)) = a$ and
  $\forall b \in B : f(h(b)) = b$, then
  f is *bijective* and $g = h = f^{-1}$. ⟨lemma 2.1 ∈ [1]⟩

- Theorems for $f : A \to B$ and $g : B \to C$: ⟨exc. 1.2, 4 p. 21 ∈ [1]⟩
  $$\begin{array}{lcl} g \circ f \textit{ injective} \text{ and } f \textit{ surjective} & \Rightarrow & g \textit{ injective} \\ g \circ f \textit{ injective} & \Rightarrow & f \textit{ injective} \\ g \circ f \textit{ surjective} & \Rightarrow & g \textit{ surjective} \\ g \circ f \textit{ surjective} \text{ and } g \textit{ injective} & \Rightarrow & f \textit{ surjective} \end{array}$$

2 Relations

FIXME: This section may not be entirely consistent. Certain texts differ in terminology and I haven't sorted out all of it.

Premises: A, B ***sets***

- **A relation** C on $A \overset{def}{=}$ a *subset* C of $A \times A$. ⟨p. 21 ∈ [1]⟩
  Writing xCy means $(x, y) \in C$. ⟨(6.2) p. 29 ∈ [3]⟩

- *Properties* that a *relation* C on a set A *can have*: ⟨(6.2) p. 29 ∈ [3]⟩
  Reflexivity: $\forall a \in A : aCa$
  Irreflexivity: $\forall a \in A : \neg(aCa)$
  Symmetry: $aCb \Rightarrow bCa$
  Asymmetry: $(aCb) \wedge (bCa) \Rightarrow a = b$
  Transitivity: $(aCb) \wedge (bCc) \Rightarrow aCc$
  Totality: $\forall a, b \in A : (a = b) \vee (aCb) \vee (bCa)$
  Partial: *Not* necessarily *total*

- **$=$ is an *equivalence relation* on** $A \overset{def}{=}$
  Reflexivity: $\forall a \in A : a = a$
  Symmetry: $(a = b) \Rightarrow (b = a)$
  Transitivity: $(a < b) \wedge (b < c) \Rightarrow a < c$
  ⟨p. 22 ∈ [1]⟩

- **$<$ is an *order relation* / *simple order* / *linear order* on** $A \overset{def}{=}$
  (Would this be called a *total order* according to ⟨(6.1) p. 29 ∈ [3]⟩?)
  Comparability: $\forall a, b \in A, a \neq b$: *either* $a < b$ or $b < a$
  Irreflexivity: $a < a$ *never holds*
  Transitivity: $(a < b) \wedge (b < c) \Rightarrow a < c$
  ⟨p. 24 ∈ [1]⟩

- According to ⟨(6.2) p. 29 ∈ [3]⟩, **an *order relation*** $< \overset{def}{=}$
  Asymmetry: $(a < b) \wedge (b < a) \Rightarrow a = b$
  Transitivity: $(a < b) \wedge (b < c) \Rightarrow a < c$
  ⟨p. 24 ∈ [1]⟩

- **$\prec$ is a *strict partial order* on** $A \overset{def}{=}$
  Irreflexivity: $a \prec a$ *never holds*
  Transitivity: $(a \prec b) \wedge (b \prec c) \Rightarrow a \prec c$
  ⟨p. 68 ∈ [1]⟩

- **Partial order** $\preceq$: $a \preceq b \overset{def}{=} a \prec b$ or $a = b$ ⟨p. 71 ∈ [1]⟩.
  The following holds: ⟨p. 187 ∈ [1]⟩
  1) $\forall \alpha : \alpha \preceq \alpha$ (reflexivity)
  2) $(\alpha \preceq \beta) \wedge (\beta \preceq \alpha) \Rightarrow \alpha = \beta$ (asymmetry)
  3) $(\alpha \preceq \beta) \wedge (\beta \preceq \gamma) \Rightarrow \alpha \preceq \gamma$ (transitivity)

- Order relations:
  1) *Lexicographic order* or *dictionary order* on $A \times B$:
     $(a_1, b_1) < (a_2, b_2) \overset{def}{=} a_1 <_A a_2$ or $(a_1 = a_2) \wedge (b_1 <_B b_2)$ ⟨p. 26 ∈ [1]⟩
  2) *Generalized dictionary order*: $(a_1, a_2, \dots) < (b_1, b_2, \dots) \overset{def}{=}$
     $\exists n \geq 1 : a_n < b_n$ and $\forall i < n : a_i = b_i$ ⟨p. 65 ∈ [1]⟩

3 Logic and Proof Techniques

- **The *converse* of** $A \Rightarrow B \stackrel{def}{=} B \Rightarrow A$. ⟨p. 8-9 ∈ [1]⟩
- **The *contrapositive* of** $A \Rightarrow B \stackrel{def}{=} \neg B \Rightarrow \neg A$. ⟨p. 8 ∈ [1]⟩

4 Set Partitions and Equivalence Classes

Premises: A, B ***sets***

- ***Partition* of** $A \stackrel{def}{=}$ ⟨p. 23 ∈ [1]⟩
  a collection of *disjoint non-empty subsets* of A whose *union* is A.
- ***Equivalence class*** $[a]$ **of** a **with respect to** $\sim \stackrel{def}{=}$
  $[a] = \{b \mid b \sim a\}$. ⟨p. 23 ∈ [1]⟩, ⟨(6.5) p. 30 ∈ [3]⟩
- *Equivalence classes* are either *distinct* or *equal.* ⟨p. 23 ∈ [1]⟩
- *Equivalence classes* on sets are *partitions* of those sets. ⟨p. 23 ∈ [1]⟩
- **The *quotient set*** $X/\sim$ **of *equivalence classes* w.r.t.** $\sim$ **on** $A \stackrel{def}{=}$
  $X/\sim = \{[a] \mid a \in A\}$. ⟨(6.5) p. 31 ∈ [3]⟩

5 The Real Numbers and Binary Operations

Premises: A, B ***sets***

- ***Binary operation* on** $A \stackrel{def}{=}$
  a *function* mapping $A \times A$ into A (e.g. $f(a_1, a_2)$ or $a_1 + a_2$). ⟨p. 30 ∈ [1]⟩
- The set $\mathbb{R}$ (the *real numbers*) *exists* with *oprations* $+, \cdot$ and *order relation* $<$. Algebraic properties $\forall a, y, z \in \mathbb{R}$:

1)	$(x+y)+z = x+(y+x)$	associativity of $+$ and $\cdot$
	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	
2)	$x + y = y + x$	commutativity of $+$ and $\cdot$
	$x \cdot y = y \cdot x$	
3)	$x + 0 = x$	identity elements for $+$ and $\cdot$
	$x \cdot 1 = x$	
4)	$\forall x \in \mathbb{R} : \exists! y \in \mathbb{R} : x + y = 0$	inverse elements for $+$ and $\cdot$
	$\forall x \in \mathbb{R} \setminus \{0\} : \exists! y \in \mathbb{R} : x \cdot y = 1$	
5)	$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$	distributivity of $+$ and $\cdot$
6)	$x > y \Rightarrow x + z > y + z$	
	$x > y \wedge z > 0 \Rightarrow x \cdot z > y \cdot z$	
7)	Relation $<$ has the *least upper bound* property	
8)	$x < y \Rightarrow \exists z \in \mathbb{R} : (x < z) \wedge (z < y)$	

  ⟨p. 30 ∈ [1]⟩
  1)-5) defines a *field.* The *laws of algebra* follows from 1)-5).
  1)-6) defines an *ordered field.* The *laws of inequality* follows by Adding 6) to 1)-5)
  7)-8) defines a *topology* - a *linear continuum.*
  Note that 1)-7) ⇒ 8). So 8) may be considered a *theorem* and 1)-7) *axioms.*

  ⟨p. 30 ∈ [1]⟩

6 Ordered Sets

6.1 Boundary Elements, Successors, Predecessors

Premises: A, B ***sets***

- For *ordered set* A and $A_0 \subset A$:
  A_0 **is *bounded above*** $\stackrel{def}{=} \exists b \in A : \forall x \in A_0 : x \leq b$.
  b **is the *upper bound* of** A_0. ⟨p. 27 ∈ [1]⟩

- For *ordered set* A and $A_0 \subset A$:
  A_0 **is *bounded below*** $\stackrel{def}{=} \exists b \in A : \forall x \in A_0 : b \leq x$.
  b **is the *lower bound* of** A_0. ⟨p. 27 ∈ [1]⟩

- For a *strict partial order* $\prec$ on A and $A_0 \subset A$:
  An *upper bound* on A_0 $\stackrel{def}{=}$
  an element $c \in A$ such that: $\forall b \in A_0 : b = c$ or $b \prec c$. ⟨p. 70 ∈ [1]⟩

- For *ordered set* A and $B \subset A$, with a *strict order* $\prec$: ⟨(6.2) p. 29 ∈ [3]⟩
  a **is the *largest element* of** B $\stackrel{def}{=} a \in B$ and $\forall x \in B : x \prec a$. ⟨p. 25 ∈ [1]⟩
  a **is the *smallest element* of** B $\stackrel{def}{=} a \in B$ and $\forall x \in B : a \prec x$. ⟨p. 25 ∈ [1]⟩

- **A *maximal element* of** A $\stackrel{def}{=}$ an element $m \in A$ such that for no element $a \in A$ does the relation $m \prec a$ hold. ⟨p. 70 ∈ [1]⟩

- For a *total order*:
  Maximal element and *largest element* is the same. ⟨(6.2) p. 29 ∈ [3]⟩

- For *ordered set* A:
  $a \in A$ **is an *immediate predecessor* of** $b \in A$ $\stackrel{def}{=} \{x \mid a < x < b\} = \emptyset$. ⟨p. 28 ∈ [1]⟩
  $b \in A$ **is an *immediate succcessor* of** $a \in A$ $\stackrel{def}{=} \{x \mid a < x < b\} = \emptyset$. ⟨p. 28 ∈ [1]⟩

6.2 Order Types, Principles, Properties, Theorems

Premises: A, B ***sets***

- **An *ordered set*** A **has the *least upper bound property*** $\stackrel{def}{=}$
  Every *non-empty* subset A_0 of A that is *bounded above* has a *least upper bound* in A ⟨p. 27 ∈ [1]⟩
  We need this property of the *real numbers* to prove things like:
  The *Archimedian ordering property*: $\mathbb{Z}_+$ has *no upper bound* in $\mathbb{R}$ ⟨p. 33 ∈ [1]⟩
  The *existence* of a *unique positive square root* $\sqrt{x}$ ⟨p. 33 ∈ [1]⟩

- **An *ordered set*** A **has the *greatest lower bound property*** $\stackrel{def}{=}$
  Every *non-empty* subset A_0 of A that is *bounded below* has a *greatest lower bound* in A ⟨p. 27 ∈ [1]⟩

- An ordered set A has the *greatest lower bound property* $\Leftrightarrow$
  A has the *least upper bound property.* ⟨exc. 1.3, 13 p. 29 ∈ [1]⟩

- A **and** B **have *same order type*** $\stackrel{def}{=}$
  $\exists f : A \to B$ *bijective* $: a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2)$. ⟨p. 25 ∈ [1]⟩

- Order types:
  Every *nonempty finite ordered* set has the *order type* of a *section* $\{1, \dots, n\}$ of $\mathbb{Z}_+$ ⟨thm. 10.1, p. 64 ∈ [1]⟩
  $\mathbb{Z}_+$, $\{1, \dots, n\} \times \mathbb{Z}_+$, $\mathbb{Z}_+ \times \mathbb{Z}_+$, $\mathbb{Z}_+ \times (\mathbb{Z}_+ \times \mathbb{Z}_+)$ all have *different order types* ⟨p. 65 ∈ [1]⟩

- **The set A with *order relation* $<$ is *well-ordered*** $\stackrel{def}{=}$
  every nonempty subset of A has a *smallest element* ⟨p. 63 ∈ [1]⟩ ⟨p. 7 (2.2) ∈ [3]⟩
  E.g. the *well-ordering property* of $\mathbb{Z}_+$ or $\mathbb{N}$ (so $\mathbb{Z}_+$ and $\mathbb{N}$ are *well-ordered*):
  $\forall A \subset \mathbb{Z}_+, A \neq \emptyset : A \textit{ has a smallest element}$ ⟨thm. 4.1 ∈ [1]⟩ ⟨p. 7 (2.2) ∈ [3]⟩

- **Well-Ordering Theorem**. There *exists* an *order relation* on *any set* which is a *well-ordering* ⟨Zermelo 1904, p. 65 ∈ [1]⟩

- **The *section* S_α of a *well-ordered* set A by $\alpha \in A$** $\stackrel{def}{=}$
  $S_\alpha = \{x \in A \mid x < a\}$ ⟨p. 66 ∈ [1]⟩
  Examples:
  S_n a *section* of $\mathbb{Z}_+ : S_n = \{x \in \mathbb{Z}_+ \mid x < n\}$ ⟨p. 32 ∈ [1]⟩
  S_{n+1} of $\mathbb{Z}_+$ *includes* $n : S_{n+1} = \{1, \dots, n\}$ ⟨p. 32 ∈ [1]⟩

- **The *minimal uncountable well-ordered* set and S_Ω :**
  There *exists* a *well-ordered* set A having a largest element Ω, such that the section S_Ω of A by Ω is *uncountable* but *every other section* of A is *countable. A has unique order-type.* ⟨lemma 10.2 p.66 ∈ [1]⟩

- $\bar{S}_\Omega \stackrel{def}{=} S_\Omega \cup \{\Omega\}$ (on the *minimal uncountable* set) ⟨p. 66 ∈ [1]⟩

- $A \underset{countable}{\subset} S_\Omega \Rightarrow A$ has an *upper bound* a, $a \in S_\Omega$.
  (on the *minimal uncountable* set) ⟨thm. 10.3 p. 66 ∈ [1]⟩

- **The maximum principle**: For a *strict partial order* $\prec$ on A there *exists* a *maximal simply ordered* subset $B \subset A$. ⟨p. 69 ∈ [1]⟩
  Similarly for a *partial order* $\preceq$. ⟨p. 71 ∈ [1]⟩

- **Zorn's lemma**: For a *strict partial order* $\prec$ on A.
  If *every simply ordered subset* of A *has* an *upper bound in* A, then A has a *maximal element.* ⟨p. 70 ∈ [1]⟩
  Similarly for a *partial order* $\preceq$. ⟨p. 71 ∈ [1]⟩

- *Well-ordering theorem* $\equiv$ *maximum principle* $\equiv$ *choice axiom* $\equiv$ *Zorn's lemma.* ⟨p. 70, p. 71, exc. 1 suppl.;6 p. 73, exc. 1 suppl.;7 p.73 ∈ [1]⟩

7 Cartesian Products, Tuples and Sequences

Premises: A, B ***sets***, C **a *non-empty collection* of *sets***

- **An *indexing function* for** $C \overset{def}{=}$
  a *surjective* function $f : J \to C$ from *some set* J, called the ***index set***, to C. ⟨p. 36 ∈ [1]⟩
- **An *indexed family of sets*** $\overset{def}{=} C$ with *indexing function* f.
  Given $\alpha \in J$, we shall denote the set $f(\alpha)$ by C_α.
  We denote the indexed family itself by $\{C_\alpha\}_{\alpha \in J}$ ⟨p. 36 ∈ [1]⟩
- $\bigcup_{\alpha \in J} C_\alpha \overset{def}{=} \{x \mid \exists \alpha \in J : x \in C_\alpha\}$ ⟨p. 36 ∈ [1]⟩
- $\bigcap_{\alpha \in J} C_\alpha \overset{def}{=} \{x \mid \forall \alpha \in J : x \in C_\alpha\}$ ⟨p. 36 ∈ [1]⟩
- Tupes of elements of A:
  - **An m-*tuple of elements of*** A, for *positive integer* $m \overset{def}{=}$
    a *function* $\vec{a} : \{1, \ldots, m\} \to A \equiv (a_1, \ldots, a_m)$ ⟨p. 37 ∈ [1]⟩
  - **An ω-*tuple of elements of*** $A \equiv$ **a *sequence*** $\equiv$ **an *infinite sequence of elements of*** $A \overset{def}{=}$
    a *function* $\vec{a} : \mathbb{Z}_+ \to A \equiv (a_1, a_2, \ldots) \equiv (a_n)_{n \in \mathbb{Z}_+}$ ⟨p. 38 ∈ [1]⟩
  - **A J-*tuple of elements of*** A, for *arbitrary index set* $J \overset{def}{=}$
    a *function* $\vec{a} : J \to A \equiv (a_\alpha)_{\alpha \in J}$ ⟨p. 113 ∈ [1]⟩
- **The i-th *coord* of** $\vec{a} \overset{def}{=} \vec{a}(i) \equiv a_i$ ⟨p. 37 ∈ [1]⟩
- Cartesian Products:
  - **Cartesian product** $\overset{def}{=} \prod_{i=1}^{m} A_i \equiv A_1 \times \cdots \times A_m$, the set of all m-*tuples* $(a_1, \ldots, a_m)$ of elements of A such that $\forall i : a_i \in A_i$. ⟨p. 37 ∈ [1]⟩
    Here, $\{A_1, \ldots, A_m\}$ is indexed by $\{1, \ldots, m\}$ and $A = A_1 \cup \cdots \cup A_m$.
  - **The *infinite cartesian product*** $\overset{def}{=}$ ⟨p. 38 ∈ [1]⟩
    $\prod_{i \in \mathbb{Z}_+} A_i \equiv A_1 \times A_2 \times \cdots$, the set of *all ω-tuples of elements of A*
    Here, $\{A_1, A_2, \ldots\}$ is indexed by $\mathbb{Z}_+$ and $A = A_1 \cup A_2 \cup \cdots$.
  - **Cartesian product of *arbitrary index set*** $J \overset{def}{=}$ ⟨p. 113, 114 ∈ [1]⟩
    $\prod_{\alpha \in J} A_\alpha$ or simply $\prod A_\alpha$, the set of *all tuples of elements of A*
    Here, $\{A_\alpha\}$ is indexed by J and $A = \bigcup_{\alpha \in J} A_\alpha$.
    The elements of $\prod A_\alpha$ are denoted (a_α) or $(a_\alpha)_{\alpha \in J}$.
- **For *cartesian products***:
  If $A = A_1 = A_2 = \cdots$, we write A^ω for *countably infinite tuples*, A^m for *finite tuples*, or A^J for *tuples with arbitrary index set.*
  E.g. $\mathbb{R}^m$ is *the Euclidean m-space.* $\mathbb{R}^\omega \equiv \vec{a} : \mathbb{Z}_+ \to \mathbb{R}$. ⟨p. 38, p. 113 ∈ [1]⟩

8 The Size of Sets

8.1 Finite Sets

Premises: A, B ***sets***, F ***finite set***

- A ***finite*** $\stackrel{def}{=}$ $A = \emptyset$ *or* $\exists f : f : A \to \{1, \ldots, n\}, f$ *bijective* ⟨p. 39 ∈ [1]⟩
- Let $n \in \mathbb{Z}_+$ and $b_0 \in A$. Then there *exists* a *bijective correspondence* f of the set $\{1, \ldots, n+1\}$ $\Leftrightarrow$ there *exists* a *bijective correspondence* g of the set $A \backslash \{b_0\}$ with $\{1, \ldots, n\}$ ⟨lemma 6.1, p. 40 ∈ [1]⟩
- Assume there *exists* a *bijection* $f : A \to \{1, \ldots, n\}$ for some $n \in \mathbb{Z}_+$. Let $B \subsetneq A$.
  Then there *exists no bijection* $g : B \to \{1, \ldots, n\}$, but if $B \neq \emptyset$, there *exists* a *bijection* $h : B \to \{1, \ldots, m\}$ for some $m < n$ ⟨thm. 6.2, p. 41 ∈ [1]⟩
- There exists *no* bijection of F with a *proper subset* of *itself* ⟨cor. 6.3 ∈ [1]⟩
- For $A \neq \emptyset$ the following are *equivalent*:
  A is *finite*
  There *exists* a *surjective function* from a *section* of $\mathbb{Z}_+$ onto A
  There *exists* an *injective function* from A onto a *section* of $\mathbb{Z}_+$
  ⟨cor. 6.7 ∈ [1]⟩
- $A \subset F \Rightarrow A$ *finite* ⟨cor. 6.6 ∈ [1]⟩
- *Finite unions* and *finite cartesian products* are *finite* ⟨cor. 6.8, p. 43 ∈ [1]⟩

8.2 Infinite Sets

Premises: A, B ***sets***, C ***countable set***, I ***infinite set***

- A ***infinite*** $\stackrel{def}{=}$ A *not finite* ⟨p. 44 ∈ [1]⟩
- A ***countably infinite*** $\stackrel{def}{=}$ there *exists* a *bijection* $f : A \to \mathbb{Z}_+$ ⟨p. 44 ∈ [1]⟩
- A ***countable*** $\stackrel{def}{=}$ A is either *finite* or *countably infinite* ⟨p. 45 ∈ [1]⟩
- A ***uncountable*** / ***overcountable*** $\stackrel{def}{=}$
  A *not countable* ⟨p. 45 ∈ [1]⟩ ⟨def. A.7 p. 151 ∈ [4]⟩
- For $A \neq 0$ the following are *equivalent*: ⟨thm. 7.1 ∈ [1]⟩
  A is *countable*
  There *exists* a *surjective function* $f : \mathbb{Z}_+ \to A$
  There *exists* an *injective function* $g : A \to \mathbb{Z}_+$
- The following are equivalent:
  A is *infinite*
  There *exists* an *injective function* $f : \mathbb{Z}_+ \to A$
  There *exists* a *bijection* of A with a *proper subset* of *itself*
  ⟨thm. 9.1, p. 57 ∈ [1]⟩
- There *exists no injective map* $f : \mathcal{P}(A) \to A$ and *no surjective map* $g : A \to \mathcal{P}(A)$ ⟨thm. 7.8 ∈ [1]⟩

- $A \underset{infinite\ subset}{\subset} \mathbb{Z}_+ \Rightarrow A$ *countably infinite* ⟨lemma 7.2 ∈ [1]⟩
- A *subset* of a *countable set* is *countable* ⟨cor. 7.3 ∈ [1]⟩
- A *countable union* of *countable sets* is *countable* ⟨thm. 7.5 ∈ [1]⟩
- A *finite product* of *countable sets* is *countable* ⟨thm. 7.6 ∈ [1]⟩
- $\{0,1\}^\omega$ is *uncountable* ⟨thm. 7.7 ∈ [1]⟩

8.3 Cardinality

Premises: A, B ***sets***, F ***finite set***

- *Cardinality* of F is 0 or $n, n > 0$ ⟨p. 39 ∈ [1]⟩
- The *cardinality* of F is *uniquely determined* by F ⟨cor. 6.5 ∈ [1]⟩
- $A \subsetneq F \Rightarrow \text{card } A < \text{card } F$ ⟨cor. 6.6 ∈ [1]⟩
- **card** $A >$ **card** $B \stackrel{def}{=}$ ⟨exc. 9;7 p. 62 ∈ [1]⟩
  there *exist* an *injection* $f : B \to A$, but *no injection* $g : A \to B$.
- **card** $A =$ **card** $B \stackrel{def}{=}$ there *exist* a *bijection* $f : A \to B$. ⟨exc. 7;6 p.51 ∈ [1]⟩
- ***Continuum Hypothesis*** $\stackrel{def}{=} \nexists a : \text{card } \mathbb{Z}_+ < a < \text{card } \mathbb{R}$. ⟨exc. 9;8 p. 62 ∈ [1]⟩
- ***Generalised Continuum Hypothesis*** **for the** ***infinite set*** **A** $\stackrel{def}{=}$
  $\nexists a : \text{card } A < a < \text{card } (\mathcal{P}(A))$ ⟨exc. 9;8 p. 62 ∈ [1]⟩
  The continuum hypotheses are *independent* of the *usual axioms* of *set theory* ⟨exc. 9;8 p. 62 ∈ [1]⟩

8.4 Induction and Recursion

- $A \underset{subset}{\subset} \mathbb{R}$ **is *inductive*** $\overset{def}{=} 1 \in A$ and $\forall x \in A : x + 1 \in A$ ⟨p. 32 ∈ [1]⟩
- $\mathbb{Z}_+$ **of *positive integers*** $\overset{def}{=} \mathbb{Z}_+ = \bigcap_{A \in \mathcal{A}} A$, where $\mathcal{A}$ is the *collection* of *all inductive subsets* of $\mathbb{R}$ ⟨p. 32 ∈ [1]⟩
- $\mathbb{Z}_+$ is *inductive* and 1 is the *smallest element* of $\mathbb{Z}_+$ ⟨p. 32 ∈ [1]⟩
- **Principle of induction**: ⟨(2.4), (2.5) p. 8 Mat 2AL ∈ [3]⟩
  If A is an *inductive set* of *positive integers*, then $A = \mathbb{Z}_+$. ⟨p. 32 ∈ [1]⟩
- ***Strong induction principle*** a.k.a. ***Complete induction principle***:
  Let A be a set of *positive integers.*
  If $\forall n \in \mathbb{Z}_+ : (S_n \subset A \Rightarrow n \in A)$, then $A = \mathbb{Z}_+$.
  ⟨thm. 4.2 p. 33 ∈ [1]⟩ ⟨(2.7), (2.8) p. 9 Mat 2AL ∈ [3]⟩
- **Principle of recursive definition**. Given a formula that defines $h(1)$ as a unique element of A, and for $i > 1$ defines $h(i)$ uniquely as an element of A in terms of the values of h for positive integers less than i, this formula determines a unique function $h : \mathbb{Z}_+ \to A$ ⟨p. 47 ∈ [1]⟩
- **Principle of recursive definition** (more general). Let $a_0 \in A$. Suppose ρ is a function that assigns, to each function f mapping a non-empty section of $\mathbb{Z}_+$ into A.
  Then there *exists* a *unique function* $h : \mathbb{Z}_+ \to A$ such that:
  $h(1) = a_0$
  $h(i) = \rho(h|_{\{1,\dots,i-1\}})$, for $i > 1$
  ⟨thm. 8.4, p. 54 ∈ [1]⟩
- **General principle of recursive definition**: Let A be a *well-ordered set* and $\mathcal{F}$ be the set of *all functions mapping sections of A onto B.*
  Given a function $\rho : \mathcal{F} \to B$, there *exists* a *unique function* $h : A \to B$ such that $\forall \alpha \in J : h(\alpha) = \rho(h|_{S_\alpha})$ ⟨exc. 1 suppl.;1 ∈ [1]⟩
- **For a *well-ordered* set J, J_0 is an *inductive subset* of J** $\overset{def}{=}$ $\forall \alpha \in J : (S_\alpha \subset J_0) \Rightarrow \alpha \in J_0$ ⟨exc. 10;7 p. 67 ∈ [1]⟩
- **Principle of Transfinite Induction** $\overset{def}{=}$ If J is a *well-ordered* set and $J_0 \underset{inductive\ subset}{\subset} J$, then $J_0 = J$ ⟨exc. 10;7 p. 67 ∈ [1]⟩
- **Axiom of Choice**. Given a *collection* $\mathcal{A}$ of *disjoint nonempty sets*, there *exists* a set B consisting of *exactly one element from each element* of $\mathcal{A}$; that is, a set B such that B is contained in the union of the elements of $\mathcal{A}$, and for each $A \in \mathcal{A}$, the set $B \cap A$ contains a single element. ⟨p.59 ∈ [1]⟩
- **Existence of a Choice Function**. Given a *collection* $\mathcal{B}$ of *nonempty* sets (not necessarily disjoint), there *exists* a function $c : \mathcal{B} \to \bigcup_{B \in \mathcal{B}} B$ such that $c(B)$ is an element of B, for each $B \in \mathcal{B}$. ⟨lemma 9.2 ∈ [1]⟩

9 Common and Well-Known Sets

9.1 Very Concrete Well-Known Sets

- About $\mathbb{Z}_+$:
  - $\mathbb{Z}_+$ is *inductive* and 1 is the *smallest element* of $\mathbb{Z}_+$ ⟨p. 32 ∈ [1]⟩
  - $\mathbb{Z}_+$ is *not finite* ⟨cor. 6.4 ∈ [1]⟩
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_+ \times \mathbb{Z}_+$ are all *countably infinite* ⟨cor. 7.4 ∈ [1]⟩ ⟨thm. A.8 p. 151 ∈ [4]⟩
- ***Integers*** $\mathbb{Z} \overset{def}{=} \mathbb{Z}_+ \cup \{0\} \cup (-\mathbb{Z}_+)$ ⟨p. 32 ∈ [1]⟩
- ***Rational numbers*** $\mathbb{Q} \overset{def}{=} \frac{a}{b}$, where $a, b \in \mathbb{Z}$ ⟨p. 32 ∈ [1]⟩

9.2 More General Well-Known Sets

Premises: A, B ***sets***

- ***Rays* on the *ordered set*** A, for $a \in A \overset{def}{=}$ ⟨p. 85 ∈ [1]⟩
  open ray: $]a, +\infty[= \{x \mid x > a\}$
  open ray: $]-\infty, a[= \{x \mid x < a\}$
  closed ray: $[a, +\infty[= \{x \mid x \geq a\}$
  closed ray: $]-\infty, a] = \{x \mid x \leq a\}$
- B **is *convex in*** A for the *ordered set* A and for $B \subset A \overset{def}{=}$
  $\forall a, b \in A$: *all of* $]a, b[$ in A *lies in* B (i.e.: $x \in]a, b[\Rightarrow x \in B$) ⟨p. 90 ∈ [1]⟩
  E.g.: *Intervals* and *rays* in *any* A are *convex* in A ⟨p. 90 ∈ [1]⟩

10 Misc

Premises: A, B ***sets***

- A ***intersects*** $B \overset{def}{=} A \cap B \neq \emptyset$ ⟨p. 96 ∈ [1]⟩

11 Methods

Premises: A, B ***sets***

- Constructing well-ordered sets:
  Finite sets are *well-ordered* using *any bijection* to the ordered set $\{1, \ldots, n\}$
  Any *subset* of a *well-ordered* set is *well-ordered* (using the restricted order relation)
  For *well-ordered* sets A, B, then $A \times B$ is *well-ordered* in the *dictionary order*
  No one has ever constructed a well-ordering of $(\mathbb{Z}_+)^\omega$ (dictionary order does *not* work) ⟨p. 65 ∈ [1]⟩
  ⟨p. 63-64 ∈ [1]⟩

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