Linear Algebra Summary

Ánoq of the Sun, Hardcore Processing *

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1 Vector Subspaces

Let V be a vector space.

- A *subspace* of V is a subset $S \subseteq V$ such that when the addition and scalar multiplication of V are used to add and scalar-multiply the elements of S, then S is a vector space (def. 1.10 [1])
- If a subset S of V satisfies:
  1. S is nonempty
  2. S is closed under addition - i.e. $a, b \in S \Rightarrow a + b \in S$
  3. S is closed under scalar multiplication - i.e. $a \in S, r \in \mathbb{R} \Rightarrow ra \in S$

  Then S is a subspace of V (subspace thm., thm. 1.11 [1])

2 Dimension Theory

Let $F = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\}$ be any finite set of vectors in a vector space V.

- A *linear combination* of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$ is any vector of the form $r_1\overrightarrow{v_1}, r_2\overrightarrow{v_2}, \ldots, r_n\overrightarrow{v_n}$, where $r_1, r_2, \ldots, r_n \in \mathbb{R}$ are called the *coefficients* (def. 3.1 [1])
- The subset of V *spanned* by F is the set of all linear combinations of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$. This is called the *span* of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$ and is denoted $span\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\}$ (def. 3.2 [1])
- The span of F in V is a subset of V (thm. 3.3 [1])
- F is *linearly independent* $\doteq$
  $r_1\overrightarrow{v_1} + r_2\overrightarrow{v_2} + \cdots + r_n\overrightarrow{v_n} = \overrightarrow{0} \Rightarrow r_1 = r_2 = \cdots = r_n = 0$.
  Otherwise F is *linearly dependent* (def. 3.4 [1])
- F is linearly dependent $\Leftrightarrow$ one of the vectors can be written as a linear combination of the other vectors in F (thm. 3.5 [1])
- F is a basis for $V \doteq F$ is linearly independent and spans all of V (def. 3.6 [1])

- V has dimension $n \doteq$ there is a basis for V containing exactly n vectors. We say that V is *n-dimensional* and we write $\dim V = n$ (def. 3.7 [1])
- V is *finite-dimensional* $\doteq$ V has a finite basis. Otherwise V is *infinite-dimensional* (def. 3.8 [1])
- F spans V and $\{w_1, w_2, \ldots, w_m\}$ is a linearly dependent subset of V $\Rightarrow n \geq m$ (comparison theorem, thm. 3.9 [1])
- F and $\{w_1, w_2, \ldots, w_m\}$ are bases for $V \Rightarrow m = n$ (thm. 3.10 [1])
- F spans $V \Rightarrow$ some subset of F is a basis for V (contraction theorem, thm. 3.11 [1])
- The free variables in row-echelon form after a Gauss-elimination of the linear system of F are the linearly dependent vectors of F (p. 118 [1])
- Let F be a linearly independent subset of V and $v_{n+1} \in V$.
  $v_{n+1} \notin$ *span* $F \Rightarrow \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ is linearly independent (expansion lemma, lemma 3.12 [1])
- Let F be a linearly independent subset of V.
  $\exists v_{n+1}, v_{n+2}, \ldots, v_{n+k} \in V : \{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+k}\}$ is a basis for V (expansion thm., thm. 3.13 [1])
- A subspace S of V is finite-dimensional and $\dim S \leq \dim V$ (thm. 3.14 [1])
- Let V be n-dimensional. F spans $V \Leftrightarrow F$ is linearly dependent (thm. 3.15 [1])
- Let F be an ordered basis for V.
  For any $\overrightarrow{v} \in V$ write $v = r_1 \overrightarrow{v_1} + r_2 \overrightarrow{v_2} + \cdots + r_n \overrightarrow{v_n}$.
  The *coordinate vector* for $\overrightarrow{v}$ with respect to F is the element $[\overrightarrow{v}]_F \in \mathbb{R}^n$:
  $[\overrightarrow{v}]_F \doteq \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$ (thm. 3.16 [1])
- Let F be an ordered basis for V. $\forall \overrightarrow{v} \in V : \exists! r_1, r_2, \ldots, r_n \in \mathbb{R} : \overrightarrow{v} = r_1 \overrightarrow{v_1} + r_2 \overrightarrow{v_2} + \cdots + r_n \overrightarrow{v_n}$ (thm. 3.17 [1])

3 Inner Product Spaces

- $|\langle v, w \rangle| \leq ||v||\ ||w||$ (Cauchy-Schwarz inequality) (thm. 4.8 [1])

4 Misc

- The *scalar product* $x \cdot y = \frac{1}{2}(||x_1||_2^2 + ||x_2||_2^2 - ||x_1 - x_2||_2^2)$ (p. 3.5 [?])
- $a + b \leq 2 \max\{a, b\}$ (p 4.4 [?])
- $ax - by = a(x - y) + (a - b)y$ (p 4.4 [?])

References

[1] Robert Messer. *Linear Algebra Gateway to Mathematics*, HarperCollins College Publishers 1994.