# Integrals Summary

Ánoq of the Sun, Hardcore Processing *

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## 1 Conventions for Calculations with Infinity

- $+\infty + a = +\infty$ for $a \in \mathbb{R} \cup \{+\infty\}$ (p. A.3 [1])
- $a \cdot (+\infty) = +\infty$ for $a > 0$ (p. A.3 [1])
- $a \cdot (+\infty) = -\infty$ for $a < 0$ (p. A.3 [1])
- $0 \cdot (\pm\infty) = 0$ (p. A.3 [1])
- $+\infty + (-\infty)$ is undefined! (p. A.3 [1])

## 2 Integrals

Rules for complex numbers, where $f, g : [a, b] \to \mathbb{C}$, $f = Re\ f + iIm\ f$:

- $\int_a^b f(x)dx \stackrel{def}{=} \int_a^b Re\ f(x)dx + i \int_a^b Im\ f(x)dx$ ⟨p. 29 ∈ [3]⟩ ⟨1.3 ∈ [1]⟩
- $Re(\int_a^b f(t)dt) = \int_a^b Re\ f(t)dt$ ⟨p. 29 ∈ [3]⟩
- $Im(\int_a^b f(t)dt) = \int_a^b Im\ f(t)dt$ ⟨p. 29 ∈ [3]⟩
- $\int_a^b \overline{f(x)}dx = \overline{\int_a^b f(x)dx}$ (complex conjugate) ⟨1.3 ∈ [1]⟩

## 3 Lebesgue Integral

### 3.1 Intuitive Description (p. A.2 [1])

Let $f \geq 0$ be a limited function on $\mathbb{R}$ with values in $[0, a[$. Divide $[0, a[$ into $[0, \frac{a}{N}[, [\frac{a}{N}, \frac{2a}{N}[, \ldots, [\frac{N-1}{N}a, a[$ and consider the orgininal sets $B_i = f^{-1}([\frac{i-1}{N}a, \frac{i}{N}a[)$, where $i = 1, 2, \ldots, N$. Let $m_1(B_i)$ be the generalized length of the set $B_i$. The number $I_N = \sum_{i=1}^{N} \frac{i-1}{N} a m_1(B_i)$ is an approximation for the area under the graph for $f$. If $\lim\limits_{N \to N} I_N$ exists then $f$ is *Lebesgue integrable* over $\mathbb{R}$ with respect to the measure $m_1$ and the value of the limit is the integral of $f$ over $\mathbb{R}$ with respect to $m_1$. Denoted $\int_{\mathbb{R}} f dm_1$, $\int_{\mathbb{R}} f(x)dx$ or $\int_{-\infty}^{\infty} f(x)dx$.

## 3.2 Measurable Sets

- Any open subset $B \subseteq \mathbb{R}$ is a countable disjoint union of open intervals: $B = \bigcup_{i=1}^{\infty} ]a_i, b_i[$, where $]a_i, b_i[ \cap ]a_j, b_j[ = \emptyset$ for $i \neq j$ (p. A.2 [1])

Let $B$ be such an open subset:

- $m_1(B) = \sum_{i=1}^{\infty}(b_i - a_i)$, where $b_i - a_i = +\infty$ if $]a_i, b_i[$ is unbounded (p. A.2 [1])
- $m_1(B) = \sum_{i=1}^{\infty}(b_i - a_i) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i) \mid B \subseteq \bigcup_{i=1}^{\infty} ]a_i, b_i[\}$, where we can even assume $]a_i, b_i[$ to be bounded (p. A.2 [1])

Borel sets $\mathbb{B}_1$ - also called measurable sets:

- There exists a least system $\mathbb{B}_1$ of subsets of $\mathbb{R}$ (which include the open sets) and a unique function (the Lebesgue measure) $m_1 : \mathbb{B}_1 \to [0, +\infty]$ with the following properties:
  1. $B \in \mathbb{B}_1 \Rightarrow \complement B \in \mathbb{B}_1$
  2. $B_1, B_2, \ldots \in \mathbb{B}_1 \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathbb{B}_1$
  3. $m_1(\emptyset) = 0$ and $m_1(]a, b[) = b - a$ for $-\infty < a < b < +\infty$
  4. $(B_1, B_2, \ldots \in \mathbb{B}_1 \wedge (i \neq j \Rightarrow B_i \cap B_j = \emptyset)) \Rightarrow$ $m_1(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} m_1(B_n)$

  These are the Borel sets - also called measurable sets (p. A.2 [1])
- Open sets, closed sets, singletons, countable sets, $\mathbb{Q}$, $\mathbb{R}$, all intervals are all in $\mathbb{B}_1$ (p. A.3 [1])
- $m_1$ can be defined by: $m_1(B) \doteq \inf\{\sum_{i=1}^{\infty}(b_i - a_i) \mid B \subseteq \bigcup_{i=1}^{\infty} ]a_i, b_i[\}$, where we can even assume $]a_i, b_i[$ to be bounded (p. A.3 [1])
- $\forall A, B \in \mathbb{B}_1 : A \subseteq B \Rightarrow m_1(A) \leq m_1(B)$ ($m_1$ is growing) (p. A.3 [1])
- $\forall A \in \mathbb{B}_1$ : A is a *zero set* $\doteq m_1(A) = 0$ (p. A.3 [1])
- $\forall \alpha \in \mathbb{R}, B \in \mathbb{B}_1 : m_1(B) = m_1(B + \alpha)$, where $\alpha + B \doteq \{\alpha + \theta \mid \theta \in B\}$ (translation invariance) (p. 2.4 [1])

## 3.3 Measurable Functions

- A function $f : \mathbb{R} \to [-\infty, +\infty]$ is measurable $\doteq$ $\forall a \in \mathbb{R} : f^{-1}([a, +\infty])$ is measurable (def. A.2 [1])
- A function $f : \mathbb{R} \to \mathbb{C}$ is measurable $\doteq$ $Ref$ and $Imf$ are measurable ($Ref$ and $Imf$ are allowed to take on the values $\pm\infty$) (def. A.2 [1])
- $f : \mathbb{R} \to [-\infty, +\infty]$ measurable $\Rightarrow$ $\forall a, b \in \mathbb{R} : f^{-1}([a, b[) = f^{-1}([a, \infty]) \setminus f^{-1}([b, \infty])$ measurable (p. A.3 [1])
- For any continuous function $f : \mathbb{R} \to \mathbb{R} : f^{-1}([a, +\infty[)$ is closed. Hence any continuous function is messurable (p. A.3 [1])
- $f, g : \mathbb{R} \to \mathbb{C}$ measurable $\Rightarrow$ $f \pm g$ and $f \cdot g$ measurable (thm. A.3 [1])
- $f, f_1, f_2, \ldots : \mathbb{R} \to \mathbb{C}$ measurable and $(f_n)$ converges pointwise towards $f$ (i.e.: $\forall x \in \mathbb{R} : f_n(x) \to f(x) \Rightarrow f$ measurable (thm. A.3 [1])

### 3.4 Lebesque Integral

- For any non-negative function $f : \mathbb{R} \to [0, +\infty]$ we define the measure of $f$ with respect to $m_1$ by:
  $\int_{\mathbb{R}} f dm_1 \doteq \lim_{N \to \infty} \sum_{i=1}^{N+1} \frac{i-1}{N} a_N m_1(B_i^N)$.
  $a_N \to +\infty$ and $\frac{a_N}{N} \to 0$ for $N \to +\infty$ (so $a_N$ could be $\sqrt{N}$).
  $B_i^N = f^{-1}([\frac{i-1}{N} a_N, \frac{i}{N} a_N[)$ for $i = 1, 2, \ldots, N$ and $B_{N+1}^N = f^{-1}([a_N, +\infty])$ (def. A.4 [1])

- $\int_{\mathbb{R}} f dm_1 \in [0, +\infty]$ (p. A.4 [1])

- $\int_{\mathbb{R}} f dm_1 < +\infty \Rightarrow f$ is (Lebesgue) integrable (p. A.4 [1])

- $f_-(x) = \begin{cases} -f(x) & \text{when } f(x) \leq 0 \\ 0 & \text{elsewhere} \end{cases}$ (p. A.4 [1])

- $f_+(x) = \begin{cases} f(x) & \text{when } f(x) \geq 0 \\ 0 & \text{elsewhere} \end{cases}$ (p. A.4 [1])

- $f_+$ and $f_-$ are measurable. $f = f_+ - f_-$. $|f| = f_+ + f_-$ (p. A.4 [1])

- Let $f : \mathbb{R} \to [-\infty, +\infty]$ be measurable. $f_+, f_-$ integrable $\Rightarrow f$ integrable.
  $\int_{\mathbb{R}} f dm_1 \doteq \int_{\mathbb{R}} f_+ dm_1 - \int_{\mathbb{R}} f_- dm_1$ (def. A.5 [1])

- Let $f : \mathbb{R} \to \mathbb{C}$ be measurable. $f$ integrable $\Leftrightarrow Ref, Imf$ integrable.
  $\int_{\mathbb{R}} f dm_1 \doteq \int_{\mathbb{R}} Ref dm_1 + i \int_{\mathbb{R}} Imf dm_1$ (def. A.5 [1])

- For any measurable function $f$ on $B$ we define $f_0$ on $\mathbb{R}$ by:
  $f_0(x) \doteq \begin{cases} f(x) & x \in B \\ 0 & x \in \mathbb{R} \backslash B \end{cases}$
  $f$ measurable $\Leftrightarrow f_0$ measurable.
  $f_0$ integrable over $\mathbb{R} \Rightarrow f$ integrable over $B$.
  $\int_B f dm_1 \doteq \int_{\mathbb{R}} f_0 dm_1$ (p. A.5 [1])

- Let $f$ on $B \in \mathbb{B}_1$ be measurable.
  $m_1(\{x \in B \mid f(x) \neq 0\}) = 0 \Rightarrow \int_B f dm_1 = 0$ (ex. A.6 [1])

- $f, g$ integrable and $f, g$ differ only in a zero set $\Rightarrow \int f dm_1 = \int g dm_1$ (ex. A.6 [1])

- The integral of $f$ over an interval: $\int_a^b f(x) dx$ for $a < b$ (ex. A.6 [1])

### 3.5 Properties of the Lebesgue Integral

Let $B \subseteq \mathbb{R}$ be measurable:

- Let $f : B \to [0, +\infty]$ be non-negative and measurable.
  $\int_B f dm_1 = 0 \Leftrightarrow m_1(\{x \in B \mid f(x) \neq 0\}) = 0$ (thm. A.7 [1])

- A statement which applies for all $x \in B$ except for a zero set is said to apply *almost everywhere* (p. A.6 [1])

- $f = g$ almost everywhere $\Leftrightarrow \int_B |f(x) - g(x)| dx = 0$ (p. A.6 [1])

- Let $a \in \mathbb{C}$. $f, g : B \to \mathbb{C}$ integrable $\Rightarrow f + g, a \cdot f$ and $|f|$ integrable.

  1. $\int_B (f+g)(x)dx = \int_B f(x)dx + \int_B g(x)dx$
  2. $\int_B a \cdot f(x)dx = a \cdot \int_B f(x)dx$
  3. $|\int_B f(x)dx| \leq \int_B |f(x)|dx$

  (p. A.6 [1])

- Let $f_1, f_2, \ldots$ be a growing sequence of non-negative functions on $B$ (i.e. $n \leq m \Rightarrow f_n \leq f_m$) which converges pointwise towards $f : B \to [0, +\infty]$ then $\int_B f_n dm_1 \nearrow \int_B f dm_1$ (Lebesgues monotony thm., thm. A.8 [1])

- Let $f_1, f_2, \ldots : B \to \mathbb{C}$ be a sequence of integrable functions. If there exists an integrable function $g : B \to [0, +\infty]$ (majorant) so that $\forall n = 1, 2, \ldots :$ $|f_n| \leq g$ and if $f_n \to f$ pointwise for $f : B \to \mathbb{C}$, then $f$ is integrable and $\int_B f_n dm_1 \to \int_B f dm_1$ (Lebesgues majorant thm., thm. A.9 [1])

- If $f : B \to \mathbb{C}$ is measurable and $g : B \to [0, +\infty]$ is integrable and $\forall n = 1, 2, \ldots : |f_n| \leq g$, then $f$ is integrable (consequence of majorant thm., p. A.7 [1])

- Given $f : B \to \mathbb{C}$. $|f|$ integrable $\Leftrightarrow f$ integrable (consequence of majorant thm., p. A.7 [1])

- For all continuous $f : [a, b] \to \mathbb{R}$ the Riemann integral equals the Lebesgue integral: $\lim\limits_{n \to \infty} \sum_{i=1}^{n} \frac{b-a}{n} f(a + \frac{i-1}{n}(b-a)) = \int_{[a,b]} f dm_1$ (ex. A.10 [1])

## 3.6 Multiple integrals

The same sentences as above holds for $\mathbb{R}^k$.

- Lebesgue measure on $I = I_1 \times I_2 \times \cdots \times I_k$: $m_k(I) \doteq m_1(I_1) \cdot m_2(I_2) \cdot \cdots \cdot m_k(I_k)$ (p. A.11 [1])

- The integral of an integrable function $f$ on a measurable set $B \subseteq \mathbb{R}^k$: $\int_B f dm_k$ or $\int_B f(x) d^k x$ (p. A.11 [1])

- Let $I \subseteq \mathbb{R}^k, J \subseteq \mathbb{R}^l$ be boxes. If $f : I \times J \to [0, +\infty]$ is non-negative and measurable then:

  1. $x \to \int_J f(x,y) d^l y$ and $y \to \int_I f(x,y) d^k x$ are measurable and non-negative on $I$ and $J$ respectively
  2. $\int_{I \times J} f(x,y) d^{k+l}(x,y) = \int_I (\int_J f(x,y) d^l y) d^k x = \int_J (\int_I f(x,y) d^k x) d^l y$ (might be $+\infty$)

  (Tonelli theorem, thm. A.15 [1])

- Let $I \subseteq \mathbb{R}^k, J \subseteq \mathbb{R}^l$ be boxes. If $f : I \times J \to \mathbb{C}$ is integrable then:

  1. $y \to f(x,y)$ is integrable over $J$ for all $x \in I$ except for a zero set and similarly for $x \to f(x,y)$ over $I$ for all $y \in J$ except for a zero set
  2. $\int_{I \times J} f(x,y) d^{k+l}(x,y) = \int_I (\int_J f(x,y) d^l y) d^k x = \int_J (\int_I f(x,y) d^k x) d^l y$

  The functions above are assigned any value (e.g. 0) in the zero sets where they are undefined (Fubini theorem, thm. A.16 [1])

### 3.7 Properties of functions

- Let $M$ be a metric space. $\forall f : M \to \mathbb{C}$ : supp $f \doteq \overline{\{x \in M \mid f(x) \neq 0\}}$ called the *support* for $f$ (def. A.13 [1])
- Let $G \subseteq \mathbb{R}$ be open and $f, g : G \to \mathbb{C}$ continuous. $f = g$ almost everywhere $\Rightarrow f = g$ (p. A.10 [1])

### 3.8 Misc

- $\forall B \subseteq \mathbb{R}$ we define the *indicator function*:
  $1_B \doteq \begin{cases} 1 & x \in B \\ 0 & x \in \mathbb{R} \backslash B \end{cases}$
  $1_B$ measurable $\Leftrightarrow B$ measurable. $\int_{\mathbb{R}} 1_B dm_1 = m_1(B)$ (ex. A.6 [1])
- The *Dirichlet function* $1_{\mathbb{Q}}$: $\int_{\mathbb{R}} 1_{\mathbb{Q}} dm_1 = 0$ (ex. A.6 [1])
- Let $B = [1, +\infty[$. For $\alpha \in \mathbb{R}, f_\alpha : x \to x^\alpha$ is continuous on $B$ and hence measurable. $f_\alpha$ is integrable over $B \Leftrightarrow \alpha < -1$ (ex. A.11 [1])
- Let $a \in \mathbb{R}, B = ]a, a+K]$ where $0 < K < +\infty$. For $\alpha \in \mathbb{R}, g_\alpha : x \to (x-a)^\alpha$ is continuous on $B$ and hence measurable. $g_\alpha$ is integrable over $B \Leftrightarrow \alpha > -1$ (ex. A.11 [1])
- Let $I = I_1 \times I_2$ and $f : I \to \mathbb{C} = f(x_1, x_2) = f_1(x_1) f_2(x_2)$ be measurable. $f$ integrable $\Leftrightarrow f_1, f_2$ integrable. Also holds for $k > 2$ (ex. A.17 [1])

## 4 Riemann Integral

- For any division $D = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ where $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ we define the values:
  1. $G_j = \sup\{f(t) \mid t \in [x_{j-1}, x_j]\}$
  2. The *upper sum* $O(f, D)$ of $f$ $O(f, D) = \sum_{j=1}^n G_j (x_j - x_{j-1})$
  3. $g_j = \inf\{f(t) \mid t \in [x_{j-1}, x_j]\}$
  4. The *lower sum* $U(f, D)$ of $f$ $U(f, D) = \sum_{j=1}^n g_j (x_j - x_{j-1})$

  (p. 1 [2])
- $\forall D, D'(divisions) : U(f, D) \leq O(f, D')$ (p. 1 [2])
- For any limited function $f : [a, b] \to \mathbb{R}$: $\sup\{U(f, D) \mid D$ is a division of $[a, b]\} \leq \inf\{O(f, D) \mid D$ is a division of $[a, b]\}$ (p. 1 [2])
- $f : [a, b] \to \mathbb{R}$ is *Riemann integrable* $\underset{def}{=}$
  $\sup\{U(f, D) \mid D$ is a division of $[a, b]\} = \inf\{O(f, D) \mid D$ is a division of $[a, b]\}$. Then $\int_a^b f(t) dt$ is this value (def. 1 [2])
- A limited function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if for every $\epsilon > 0$ there exists a division $D$ of $[a, b]$ such that $O(f, D) - U(f, D) \leq \epsilon$ (lemma 2 [2])
- Any continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable (thm. 3 [2])

## References

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