Hilbert Spaces Summary

Ánoq of the Sun, Hardcore Processing *

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1 Some vector space names

- $\mathcal{F}(A,B)$: The set of functions $f : A \to B$
- $\mathcal{B}(A,B)$: The set of limited functions $f : A \to B$
- $\mathcal{B}(A)$: The set of limited functions $f : A \to A$
- $\mathcal{C}(A,B)$: The set of continuous functions $f : A \to B$
- $\mathcal{C}_0(M)$: Continous functions on M with compact support (p. A.10 [2])
- $l^0(\mathbb{N})$: The set of complex sequences $(x_n)_{n\in\mathbb{N}}$ where $\exists N \in \mathbb{N} : \forall n \geq N : x_n = 0$ (i.e. x_n is 0 from a certain point) (1.4 [2])
- $l^2(\mathbb{N})$: The set of squared summable complex sequences $(a_n)_{n\in\mathbb{N}}$. I.e. $\sum_{n=1}^{\infty} |a_n|^2 < +\infty$ (p. 1.9 [2])
- Let $B \subseteq \mathbb{R}$ be measurable. For $p = 1, 2$:
  $\mathcal{L}_p(B) \doteq \{f : B \to \mathbb{C} \mid f \text{ measurable and } \int_B |f|^p dm_1 < +\infty\}$ (p. A.9 [2])
- $\mathcal{L}_1(B)$ is also called $\mathcal{L}(B)$ (p. A.9 [2])
- $\mathcal{L}_2(B)$: The set of squared integrable complex functions on B (p. A.9 [2])
- $p = 1, 2$: $\mathcal{L}_{p,per}$: The set of complex functions on $\mathbb{R}$ with period 2π which are integrable (resp. squared integrable) on $[-\pi, \pi]$ (p. 2.4 [2])
- $\mathcal{C}_{per}$: 2π-periodic continuous complex functions on $\mathbb{R}$ (p. 2.4 [2])
- $n = 1, 2, \ldots, \infty : \mathcal{C}^n_{per}$: 2π-periodic n times continuous differentiable complex functions on $\mathbb{R}$ (p. 2.4 [2])
- $\mathcal{S}(\mathbb{R})$: The set of *Schwartz functions* (def. 6.14 [2])
- Let $B \subseteq \mathbb{R}$ be measurable. $p = 1, 2 : L_p(B) \doteq \{\widetilde{f} \mid f \in \mathcal{L}_p(B)\}$ where
  $\widetilde{f} \doteq \{g : B \to \mathbb{R} \mid g \text{ measurable, } g = f \text{ almost everywhere}\}$ (p. A.9 [2])
- $L_2([-\pi, \pi], \frac{1}{2\pi})$: equivalence classes of squared integrable complex functions on $[-\pi, \pi]$ with the inner product scaled by $\frac{1}{2\pi}$ (p. 2.1 [2])
- Results on $L_2([-\pi, \pi])$ can be generalized to results on $L_2([a, b])$ by translating and scaling $x \in [-\pi, \pi]$ into $x' \in [a, b]$ using: $x' = a + \frac{a-b}{2\pi}(x + \pi)$ (p. 2.2 [2])

1.1 Subspace relations

- $\mathcal{C}(A,B) \subseteq \mathcal{F}(A,B)$
- $l^0(\mathbb{N}) \subseteq \mathcal{F}(\mathbb{N},\mathbb{C})$ (p. 1.4 [2])
- $l^0(\mathbb{N}) \subseteq l^2(\mathbb{N}) \subseteq \mathcal{F}(\mathbb{N},\mathbb{C})$ (p. 1.9-1.10 [2])
- For $p = 1,2 : \mathcal{L}_p(B) \subseteq \mathcal{F}(B,\mathbb{C})$ (p. A.9 [2])
- $\mathcal{L}_2([-\pi,\pi]) \subseteq \mathcal{L}_1([-\pi,\pi])$ (p. 2.3 [2])
- $C_0^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$ (rem. 6.15 [2])
- $C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ (p. 6.17 [2])
- Let $G \underset{subset}{\subseteq} \mathbb{R}$ be open. $p = 1,2 : \mathcal{C}_0(G) \underset{subspace}{\subseteq} \mathcal{L}_p(B)$ (p. A.10 [2])
- Let $G \underset{subset}{\subseteq} \mathbb{R}$ be open. $p = 1,2 : \mathcal{C}_0(G) \underset{subspace}{\subseteq} L_p(B)$ (p. A.10 [2])
- $\forall A \underset{subset}{\subseteq} E : A^\perp \underset{subspace}{\subseteq} E$ in $(E, \langle\cdot,\cdot\rangle)$ (1.4 [2])

1.2 Condensed subspace relations

- Let $G \underset{subset}{\subseteq} \mathbb{R}$ be open. $p = 1,2 : \mathcal{C}_0^\infty$ is a condensed subset of $L_p(G)$. Furthermore: $\forall f \in \mathcal{L}_1(G) \cap \mathcal{L}_2(G) : \forall \epsilon > 0 : \exists g \in \mathcal{C}_0^\infty(G) : ||f-g||_1 < \epsilon \wedge ||f-g||_2 < \epsilon$ (thm. A.14 [2])

1.3 Completions

- $G \underset{open\ subset}{\subseteq} \mathbb{R}$. $p = 1,2 : L_p(G)$ is a completion of $\mathcal{C}_0^\infty(G)$ w.r.t. $||\cdot||_p$ (p. A.11 [2])

2 Inner Product Spaces

2.1 Definition: Inner Product Space (1.1 [2])

Let E be a vector space over $\mathbb{L}$ (i.e. $\mathbb{R}$ or $\mathbb{C}$). An *inner product* is a function $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{L}$ satisfying the following:

1. $\forall x \in E \backslash \{\overrightarrow{0}\} : \langle x,x\rangle > 0$ (positive definite)
2. $\forall x,y \in E : \langle x,y\rangle = \overline{\langle y,x\rangle}$
3. $\forall x,y \in E :: \langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$ (linear)
4. $\forall \lambda \in \mathbb{L} : \forall x,y \in E : \langle \lambda x,y\rangle = \lambda\langle x,y\rangle$ (linear in first variable)

The pair $(E, \langle\cdot,\cdot\rangle)$ is an *inner product space.*

2.2 Identities for the inner product (1.2 - 1.3 [2])

- $\forall \lambda \in \mathbb{L} : \forall x, y \in E : \langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$ (sesquelinear)
- $\forall x \in E : \langle \overrightarrow{0}, x \rangle = 0$
- $\forall x \in E : \langle x, x \rangle \Leftrightarrow x = \overrightarrow{0}$
- $\forall x, y \in E : \langle x, y \rangle + \langle y, x \rangle \in \mathbb{R}$ which also means $Im(\langle x, y \rangle) = -Im(\langle y, x \rangle)$
- $\forall x, y \in E : i\langle x, y \rangle - i\langle y, x \rangle \in \mathbb{R}$ which also means $Re(\langle x, y \rangle) = Re(\langle y, x \rangle)$
- $\forall x, y \in E : ||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$ (parallelogram identity)
- $\forall x, y \in E$ (only over $\mathbb{C}$) : $\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k ||x + i^k y||^2$ (polarization identity)

2.3 Default inner products

- $\mathbb{C}^k$: $\langle (x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots, y_k) \rangle = x_1 \overline{y_1} + x_2 \overline{y_2} + \ldots + x_k \overline{y_k}$ (1.3 [2])
- $C([a, b], \mathbb{C})$: $\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$ (1.3 [2])
- $l^0(\mathbb{N})$: $\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$. Naturally this sum is finite (1.4 [2])
- $l^2(\mathbb{N})$: $\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$ (1.10 [2])
- $B \subseteq \mathbb{R}$ measurable. $L_2(B)$: $\langle f, g \rangle = \int_B f(x) \overline{g(x)} dx$, $f, g \in L_2(B)$ (p. A.10 [2])
- $L_2([-\pi, \pi], \frac{1}{2\pi})$: $\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$, $f, g \in L_2([-\pi, \pi])$ (p. 2.1 [2])
- $k = 1, 2, \ldots : L_2([-\pi, \pi]^k)$: $\langle f, g \rangle = \frac{1}{(2\pi)^k} \int_{[-\pi, \pi]^k} f(x) \overline{g(x)} dx$ (p. 2.12 [2])

2.4 Other common inner products (1.3 [2])

- $C([a, b], \mathbb{C})$: $\langle f, g \rangle_\rho = \int_a^b \rho(x) f(x) \overline{g(x)} dx$ where $\rho \in C([a, b],]0, \infty[)$
- $L_2([a, b], \rho)$: $\langle f, g \rangle_\rho = \int_a^b f(x) \overline{g(x)} \rho(x) dx$, $f, g \in L_2([a, b])$ (p. A.10 [2])

2.5 Norms and normed product spaces

- $||x|| = \sqrt{\langle x, x \rangle}$ uniquely defines a norm on the inner product space E. The pair $(E, || \cdot ||)$ then defines a *normed vector space* (Sentence 1.6 [2])
- $C([a, b], \mathbb{C})$: Metric space norm equivalence holds for $|| \cdot ||$ and $|| \cdot ||_\rho$, where $|| \cdot || = \sqrt{\langle \cdot, \cdot \rangle}$ and $|| \cdot ||_\rho = \sqrt{\langle \cdot, \cdot \rangle}_\rho$ (p. 1.3-1.4 [2])
- $L_2([a, b])$: Metric space norm equivalence holds for $|| \cdot ||_2$ and $|| \cdot ||_\rho$, where $|| \cdot ||_2 = \sqrt{\langle \cdot, \cdot \rangle}$ and $|| \cdot ||_\rho = \sqrt{\langle \cdot, \cdot \rangle}_\rho$ (p. A.10 [2])

2.6 Default norms

- $p = 1, 2 : \mathcal{L}_p(B)$: $||f||_p = (\int_B |f(x)|^p dx)^{\frac{1}{p}}$ (N1 doesn't hold) (p. A.9 [2])
- $p = 1, 2 : L_p(B)$: $||f||_p = (\int_B |f(x)|^p dx)^{\frac{1}{p}}$ (the p-norm) (p. A.9 [2])

2.7 Continuity of the inner product

The following are equivalent:

- $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{L}$ is continuous w.r.t. the product metric given by the norm on E (p. 1.7 [2])
- For arbitrary sequences (x_n) and (y_n) in E :
  $x_n \to x_0, y_n \to y_0 \Rightarrow \langle x_n, y_n \rangle \to \langle x_0, y_0 \rangle$ for $n \to \infty$ (p. 1.8 [2])

The following holds:

- $\forall A \subseteq E : A^\perp = \overline{A}^\perp$ (p. 1.8 [2])
- $\forall A \subseteq E : A^\perp = (span\ A)^\perp = (\overline{span\ A})^\perp$ (eq. (18) p. 1.8 [2])
- $\forall A \subseteq E : A^\perp$ is a *closed subset of* E (p. 1.8 [2])

We call $\overline{span\ A}$ the *closed subset spanned by* A (p. 1.8 [2]).

3 Orthogonality in $(E, \langle \cdot, \cdot \rangle)$

- $\forall x, y \in E : \langle x, y \rangle = 0 \equiv x \perp y$. x and y are orthogonal (1.4 [2])
- $\forall x \in E : x$ is orthogonal to a set $A \subseteq E, x \perp A \Leftrightarrow \forall a \in A : x \perp a$ (1.4 [2])
- $A^\perp \doteq \{x \in E \mid \forall y \in A : \langle x, y \rangle = 0\}$. A's orthogonal complement (1.4 [2])
- $A^\perp = (spanA)^\perp$ (1.4 [2])

3.1 The following holds if $(x_i)_{i \in I}$ is a family of vectors from E where I is an arbitrary index set (1.4 [2])

- (x_i) is an *orthogonal system* $\doteq i \neq j \Rightarrow \langle x_i, x_j \rangle = 0$
- (x_i) is an *orthonormal system* $\doteq (x_i)$ is orthogonal and $\forall i \in I : ||x_i|| = 1$
- (x_i) is *linearly independent* $\doteq$ Any finite subset of (x_i) is linearly independent
- $((x_i)$ is an orthogonal system and $\forall i \in I : x_i \neq 0)$
  $\Rightarrow (x_i)$ is a liniearly independent system (Lemma 1.3 [2])

3.2 The following holds if $(x_1, x_2, \dots, x_n)$ is a finite orthogonal system in E

- $|| \sum_{i=1}^n x_i ||^2 = \sum_{i=1}^n ||x_i||^2$ (Sentence 1.4 [2])

3.3 The following holds if $(e_1, e_2, \dots, e_n)$ is a finite orthonormal system in E

- $\forall x \in E : \exists! u \in span\{e_1, e_2, \dots, e_n\} : x - u \in \{e_1, e_2, \dots, e_n\}^\perp$. $u = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$ (Projection theorem, thm. 1.5 [2])
- $u = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$ is the *orthogonal projection* of x onto the subspace $span\{e_1, e_2, \dots, e_n\}$. u has smallest possible distance (within the subspace $span\{e_1, e_2, \dots, e_n\}$) to x w.r.t. $|| \cdot ||$. (page 1.6 [2])
- $\forall x \in E : ||\langle x, e_i \rangle e_i||^2 = |\langle x, e_i \rangle|^2$ (page 1.6 [2])
- $\forall x \in E : \sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \leq ||x||^2$ (Bessels inequality p. 1.6 [2])
- $\forall x, y \in E : |\langle x, y \rangle| \leq ||x|| \, ||y||$ (the Cauchy-Schwartz inequality p. 1.7 [2])

4 Convergent series

Let $x_n \in E$ and E be a normed vector space.

- The series $\sum_{n=1}^{\infty} x_n$ is konvergent with sum $x = \sum_{n=1}^{\infty} x_n$ in $E \doteq$ The prefix sequence $(s_k)_{k \in \mathbb{N}}$ given by $s_k = \sum_{n=1}^{k} x_n$ konverges towards x for $k \to \infty$ (Definition 1.7 [2])
- Let $\sum_{n=1}^{\infty} f_n$ be a series in $\mathcal{B}(M, \mathbb{C})$. A series of constants $\sum_{n=1}^{\infty} b_n$ where $b_n \geq 0$ for $n \in \mathbb{N}$ is said to be a *majorant series* for $\sum_{n=1}^{\infty} f_n$ if $\forall n \in \mathbb{N} : ||f_n||_u \leq b_n$, i.e. if $\forall x \in M, n \in \mathbb{N} : |f_n(x)| \leq b_n$ (def. B.1 [2])
- If the series $\sum_{n=1}^{\infty} f_n$ in $\mathcal{B}(M, \mathbb{C})$ has a convergent majorant series then it converges uniformly in $\mathcal{B}(M, \mathbb{C})$ (majorant criterion, thm. B.2 [2])
- Let M be a metric space and let $\sum_{n=1}^{\infty} f_n$ be a series whose elements are limited continuous functions on M. If $\sum_{n=1}^{\infty} f_n$ has a convergent majorant series then it converges uniformly in M with continuous base function (cor. B.3 [2])

4.1 The following holds for any konvergent series $\sum_{n=1}^{\infty} x_n$ in E and all $y \in E$

- $\langle \sum_{n=1}^{\infty} x_n, y \rangle = \lim_{k \to \infty} \langle \sum_{n=1}^{k} x_n, y \rangle = \lim_{k \to \infty} \sum_{n=1}^{k} \langle x_n, y \rangle = \sum_{n=1}^{\infty} \langle x_n, y \rangle$ (eq. (16) p. 1.8 [2])
- $\langle y, \sum_{n=1}^{\infty} x_n \rangle = \sum_{n=1}^{\infty} \langle y, x_n \rangle$ (eq. (17) p. 1.8 [2])

5 Hilbert Spaces

5.1 Finite dimensional inner product spaces (p. 1.9 [2])

If E is a finite dimensional inner product space, then the following holds:

- Any finite dimensional inner product space E has orthonormal bases

- Let $(e_1, e_2, \dots, e_n)$ be an orthonormal basis. $\overrightarrow{x} = (x_1, x_2, \dots, x_n) \in \mathbb{L}^n$ is the coordinate set for the vector $x \in E, x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$
- $||x|| = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}$
- $(x_1, x_2, \dots, x_n) \to x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ is a vector space isomorphism and metric space isometrism. It's inverse is $x \to (\langle x, e_1 \rangle, \langle x, e_2 \rangle, \dots, \langle x, e_n \rangle)$
- E is complete (using the above isometrism and $\mathbb{L}^n$ is complete)

Any finite dimensional subspace of an inner product space is closed. (p. 1.11 [2])

5.2 Hilbert Spaces

- A *Hilbert space* is a complete inner product space (Def. 1.8 [2])
- Any finite dimensional inner product space is a Hilbert space (Ex. 1.9 [2])
- $l^2(\mathbb{N})$ is a Hilbert space ($l^0(\mathbb{N})$ is *not*, since it isn't complete) (p. 1.10 [2])
- $\mathcal{C}([a,b])$ with the inner product $\langle \cdot, \cdot \rangle_\rho$ is *not* a Hilbert space since it isn't complete (p. 1.10 [2])
- $B \subseteq \mathbb{R}$ be measurable. $p = 1, 2 : L_p(B)$ are Hilbert spaces (thm. A.12 [2])
- $L_2([a,b], \rho)$ is a Hilbert space (p. A.10 [2])
- $L_2([-\pi, \pi], \frac{1}{2\pi})$ is a Hilbert space (p. 2.1 [2])

5.3 Default Orthonormal Bases

- $l^2(\mathbb{N}) : (e_i)_j = \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$
  $span\{e_i \mid i \in \mathbb{N}\} = l^0(\mathbb{N})$, which is a condensed subset of $l^2(\mathbb{N})$ (ex. 1.12 p. 1.13 [2])
- For I finite or countable: $l^2(I) : (\epsilon_i)_j = \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}$ (p. 6.3 [2])

5.4 Other Common Orthonormal Bases

- $L_2([-\pi, \pi], \frac{1}{2\pi}) : (e_n)_{n \in \mathbb{Z}}$ where $e_n(x) = e^{inx}$ (thm. 2.9 [2])
- $L_2([-\pi, \pi], ?) \ : \ (1, \sqrt{2}\cos x, \sqrt{2}\sin x, \sqrt{2}\cos 2x, \sqrt{2}\sin 2x, \dots)$ (thm. 2.9 [2])
- $L_2([-\pi, \pi]^k)$: For $n = (n_1, \dots, n_k) \in \mathbb{Z}^k, \theta = (\theta_1, \dots, \theta_k) \in [-\pi, \pi]^k$: $(e_n)_{n \in \mathbb{Z}}$ where $e_n(\theta) = e^{in \cdot \theta} = e^{i(n_1\theta_1 + \cdots + n_k\theta_k)} = e^{in_1\theta_1} \cdot \dots \cdot e^{in_k\theta_k}$ (thm. 2.11 [2])

5.5 Hilbert Space Theorems

If H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, then the following holds:

- X with the restriction of $\langle \cdot, \cdot \rangle$ to $X \times X$ is an inner product space. It is also a Hilbert space $\Leftrightarrow$ X is a closed subspace of H (p. 1.11 [2])
- All finite dimensional subspaces of H are Hilbert spaces (p. 1.11 [2])
- Let $(x_i)_{i \in \mathbb{N}}$ be an orthogonal system in H. $\sum_{i=1}^{\infty} x_i$ converges in $H \Leftrightarrow \sum_{i=1}^{\infty} x_i < +\infty$ (thm. 1.10 [2])
- $\sum_{i=1}^{\infty} x_i$ converges in $H \Rightarrow ||\sum_{i=1}^{\infty} x_i||^2 = \sum_{i=1}^{\infty} ||x_i||^2$ (thm. 1.10 [2])
- <u>An orthonormal</u> basis for H is an orthonormal system $(e_i)_{i \in I}$ in H so that $\overline{span\{e_i \mid i \in I\}} = H$ (def. 1.11 p. 1.12 [2])
- Any orthonormal basis $(e_i)_{i \in I}$ is a maximal orthonormalsystem in H. I.e. there doesn't exist any vector $e \in H$ with $||e|| = 1$ so that e with $(e_i)_{i \in I}$ is an orthonormal system (p. 1.12 [2])
- An orthonormal system $(e_i)_{i \in I}$ is an orthonormal basis for $\overline{span\{e_i \mid i \in I\}}$ (p. 1.12 [2])
- Any Hilbert space has an orthonormal basis (p. 1.12 [2])
- When H is complex: Let $n \in \mathbb{N}, a \in \mathbb{C}$ so that $a^n = 1$ and $a^2 \neq 1$. $\forall x, y \in H : \langle x, y \rangle = \frac{1}{n} \sum_{k=0}^{n-1} a^k ||x + a^k y||^2$ (generalized polarization identity) (exc. 1.12 [2])
- Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis for H. $\forall x, y \in H : \langle x, y \rangle = \sum_{i=1}^{\infty} \langle x, e_i \rangle \overline{\langle y, e_i \rangle}$ (generalized Parseval equation exc. 1.10 [2])

5.6 The following holds for any Hilbert space H

The references only prove this for separable infinite dimensional Hilbert spaces. Let $(e_i)_{i \in \mathbb{N}}$ be an infinite dimensional orthonormal system in H.

- $\sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2 \leq ||x||^2$ (generalized Bessel inequality) (p. 1.13 [2])
- The series $\sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ konverges in H. (p. 1.14 [2])
- $\forall x \in H : \exists! u \in \overline{span\{e_i \mid e \in \mathbb{N}\}} : x - u \in \{e_i \mid i \in \mathbb{N}\}^{\perp}, u = \sum_{i \in \mathbb{N}} \langle x, e_i \rangle e_i$ and does not depend on the order of summation (thm. 1.13 [2])

The following are equivalent for an orthonormal system $(e_i)_{i \in \mathbb{N}}$ in H:

1. $(e_i)_{i \in \mathbb{N}}$ is an orthonormal basis for H (thm. 1.14 [2])
2. $\{e_i \mid i \in \mathbb{N}\}^{\perp} = \{\overrightarrow{0}\}$ (thm. 1.14 [2])
3. Orthonormal development: $\forall x \in H : x = \sum_{i \in \mathbb{N}} \langle x, e_i \rangle e_i$ (thm. 1.14 [2])
4. Parseval equation: $\forall x \in H : ||x||^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2$ (thm. 1.14 [2])

Let X be a closed subspace of H.

- $\forall x \in X : \exists! u \in X, v \in X^\perp : x = u + v$ (projection theorem, thm. 1.15 [2])
- X and $X^\perp$ are complementary and $H = X \bigoplus X^\perp$ (p. 1.15 [2])
- $X = X^{\perp\perp}$ (p. 1.16 [2])
- $X^\perp$ is a closed subset (p. 1.16 [2])
- Orthonormal bases for X and $X^\perp$ together will be an orthonormal basis for H (p. 1.16 [2])

Furthermore, when $(e_i)_{i\in I}$ is an orthonormal basis for X, $u = \sum_{i\in I}\langle x, e_i\rangle$ is the orthogonal projection of x onto X, which is the unique vector with smallest distance to x in X. (p. 1.15 [2]) Similarly v is the orthogonal projection of x onto $X^\perp$. (p. 1.16 [2])

5.7 Separable Hilbert Spaces

- A Hilbert space is is separable $\Leftrightarrow$ it has a finite or countable orthonormal basis (thm. 1.16 p. 1.16 [2])

6 Misc about functions

- A function f on a compact interval $[a, b]$ is said to be *peicewise continuous* if $[a, b]$ can be divided into $a = t_0 < t_1 < t_2 < \cdots < t_m = b$ where for $i = 0, 1, \ldots, m-1 :]t_i, t_{i+1}[$ equals a continuous function on $[t_i, t_{i+1}]$. This is the same as f being continuous on $i = 0, 1, \ldots, m-1 :]t_i, t_{i+1}[$ and having finite limit values $i = 0, 1, \ldots, m-1 : f(t_i+)$ from right and $i = 0, 1, \ldots, m : f(t_i-)$ from left. The same holds for a periodic function if it holds on it's restriction to one period (p. 2.7-2.8 [2])
- A function 2π-periodic function f on $\mathbb{R}$ is said to be *peicewise smooth* or *piecewise* C^1 if f is differentiable at all points in $[-\pi, \pi]$ except for a finite number of points, such that f' is piecewise continuous (we assign any value to f' where it is undefined) (p. 2.10 [2])

7 Fourier series

The subspace relation $\mathcal{L}_2([-\pi, \pi]) \subseteq \mathcal{L}_1([-\pi, \pi])$ implies that anything which holds for $f \in \mathcal{L}_1([-\pi, \pi])$ also holds for $f \in \mathcal{L}_2([-\pi, \pi])$.

- $e_n(x) \doteq e^{inx}, x \in [-\pi, \pi]$ (p. 2.1 [2])
- $c_n(f) \doteq \langle f, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ (nth Fourier coefficient, p. 2.2 [2])
- *Fourier series* or *trigonometric series* for $f \in \mathcal{L}_1([-\pi, \pi])$ or $f \in L_1([-\pi, \pi])$: $\sum_{n\in\mathbb{Z}}\langle f, e_n\rangle e_n = \sum_{n\in\mathbb{Z}} c_n(f)e^{inx}$, also written as: $f(x) \sim \sum_{n\in\mathbb{Z}} c_n(f)e^{inx}$ (p. 2.2 [2])

- Fourier series for $f \in L_2([-\pi,\pi]^k)$: $\sum_{n\in\mathbb{Z}^k} c_n(f) e^{in\cdot\theta}$ where: $c_n(f) \doteq \langle f, e_n\rangle = \frac{1}{(2\pi)^k}\int_{[-\pi,\pi]^k} f(\theta)e^{-in\cdot\theta} dm_k(\theta)$ (p. 2.13 [2])

- $\forall f, g \in \mathcal{L}_1([-\pi,\pi]) : f = g$ almost everywhere $\Rightarrow f$ has the same Fourier series as g (p. 2.3 [2])

- The Fourier series for $f \in L_2([-\pi,\pi])$ converges in $L_2([-\pi.\pi])$ (p. 2.3 [2])

- $\forall f \in L_2([-\pi,\pi]) : f = \sum_{n\in\mathbb{Z}} c_n(f)e^{inx}$ (p. 2.3 [2])

- Let $f \in \mathcal{L}_2([-\pi,\pi])$, $S_N(x) = \sum_{n=-N}^{N} c_n(f)e^{inx}$. The sequence $(S_N)_{N\in\mathbb{N}}$ converges at x in $\mathbb{C} \Rightarrow \sum_{n\in\mathbb{Z}} c_n(f)e^{inx}$ converges at x in $\mathbb{C}$. Then the sum of the Fourier series is denoted $\sum_{n\in\mathbb{Z}} c_n(f)e^{inx}$ (p. 2.3 [2])

- The Fourier series for f converges pointwise on $[-\pi,\pi[\ \Rightarrow$ the Fourier series converges $\forall x \in \mathbb{R}$ and the sum is given by the unique periodic extension of the sum to $\mathbb{R}$ (p. 2.4 [2])

- $p = 1, 2$ the Fourier series on f is the Fourier series of the restriction of f to $[-\pi,\pi]$ so $c_n(f) = c_n(f|_{[-\pi,\pi]})$ (p. 2.4 [2])

- $\forall \alpha \in \mathbb{R}, f \in \mathcal{L}_{1,per} : \int_{-\pi}^{\pi} f(\theta)d\theta = \int_{-\pi+\alpha}^{\pi+\alpha} f(\theta)d\theta = \int_{-\pi}^{\pi} f(\theta+\alpha)d\theta$ (p. 2.4 [2])

7.1 Pointwise Convergence of Fourier Series

- $\forall f \in \mathcal{L}_1([-\pi,\pi]) : c_n(f) \to 0$ for $n \to \pm\infty$ (Riemann-Lebesgue lemma, thm. 2.1 [2])

- Let $f \in \mathcal{L}_{1,per}$. $g(\theta) = \frac{f(\theta_0+\theta)+f(\theta_0-\theta)-2s}{\theta}, \theta \in]0,+\infty[$ is integrable over $]0,\delta]$ for some $\delta \Rightarrow$ the Fourier series for f converges in $\theta_0 \in \mathbb{R}$ with the sum s (Dini test, thm. 2.2 [2])

- Let $f \in \mathcal{L}_{1,per}$. $g(\theta) = \frac{f(\theta_0+\theta)+f(\theta_0-\theta)-2s}{\theta}, \theta \in]0,+\infty[$ is integrable over $]0,\delta]$ for some $\delta \Rightarrow g$ is integrable over any compact interval $[a,b]$ for $a > 0$ (remark. 2.3 [2])

The following holds for $\theta_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and piecewise continuous:

- If these limits exist: $f'_+(\theta_0) = \lim_{\theta\to 0+} \frac{f(\theta_0+\theta)-f(\theta_0+)}{\theta}, f'_-(\theta_0) = \lim_{\theta\to 0-} \frac{f(\theta_0+\theta)-f(\theta_0-)}{\theta} \Rightarrow$ the Fourier series for f converges in θ_0 towards $s = \frac{1}{2}(f(\theta_0+) + f(\theta_0-))$. If f also converges in θ_0 then the Fourier series for f converges in θ_0 towards $f(\theta_0)$ (cor. 2.4 [2])

- $\lim_{x\to\theta_0-} f'(x), \lim_{x\to\theta_0+} f'(x)$ exists $\Rightarrow f'_+(\theta_0) = \lim_{\theta\to 0+} \frac{f(\theta_0+\theta)-f(\theta_0+)}{\theta}, f'_-(\theta_0) = \lim_{\theta\to 0-} \frac{f(\theta_0+\theta)-f(\theta_0-)}{\theta}$ exists (Mikael Rørdam's lecture)

- If f is Hölder continuous of order $\alpha > 0$ in θ_0, i.e. $\exists M \geq 0, d > 0 : |\theta - \theta_0| \Rightarrow |f(\theta) - f(\theta_0)| \leq M|\theta - \theta_0|^\alpha$, then the Fourier series for f converges in θ_0 towards $f(\theta_0)$ (cor. 2.5 [2])

7.2 Uniform Convergence of Fourier Series

The following holds when $f \in C_{per}$ is piecewise smooth:

- $c_n(f') = inc_n(f)$ (lemma 2.7 [2])
- Fourier series for f converges uniformly on $\mathbb{R}$ towards f (thm. 2.8 [2])
- $\forall f \in L_2([-\pi,\pi]) : \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \sum_{n\in\mathbb{Z}} |c_n(f)|^2$ (Parseval, p. 2.12 [2])

8 Fourier Transform

Let H_1, H_2 be Hilbert spaces over $\mathbb{L}$ ($\mathbb{R}$ or $\mathbb{C}$) and $U : H_1 \to H_2$ be a limited operator:

- U is said to be *orthogonal* (in $\mathbb{R}$) or *unitary* (in $\mathbb{C}$) if it is bijective and $U^{-1} = U^*$. This is the same as when $U^*U = I_{H_1}$ and $UU^* = I_{H_2}$ where $i = 1,2 : I_{H_i}$ is the identity operator on H_i (def. 6.1 [2])
- The following are equivalent:
  1. U is unitary / orthogonal
  2. U is surjective and $U^*U = I_{H_1}$
  3. U is surjective and $\forall x, y \in H_1 : \langle Ux, Ux\rangle = \langle x, y\rangle$
  4. U is surjective and isometric, i.e. $\forall x \in H_1 : ||Ux|| = ||x||$
  5. U maps orthonormal bases for H_1 into orthonormal bases for H_2 (thm. 6.2 [2])
- Let $(e_i)_{i\in I}$ be an orthonormal basis for H_1 and $(f_i)_{i\in I}$ an orthonormal basis for H_1. There exists exactly one limited linear operator U so that $\forall i \in I : Ue_i = f_i$. U is unitary and is given by $U(\sum_{i\in I} a_i e_i) = \sum_{i\in I} a_i f_i$ for $\sum_{i\in I} |a_i|^2 < \infty$. This means that U takes the vector $x \in H_1$ into $y \in H_2$ and keeps the same coordinate set $(a_i)_{i\in I}$. We have that $\forall x \in H_1 : x = \sum_{i\in I}\langle x, e_i\rangle e_i$ and $y = Ux = \sum_{i\in I}\langle x, e_i\rangle f_i$ (p. 6.3 [2])

Let H be a separable Hilbert space over $\mathbb{L}$ and $(e_i)_{i\in I}$ an orthonormal basis for H (so I is finite or countable).

- We define a *multiplication operator* M_a on $l_2(I)$ by: $M_a(x_i)_{i\in I} = (a_i x_i)_{i\in I}$ for $(x_i)_{i\in I} \in l_2(I)$ (p. 6.4 [2])
- An operator $A \in \mathcal{B}(H)$ is *diagonalizable* $\Leftrightarrow$ there exists a unitary operator $U : H \to l_2(I)$ (where I is finite or countable) and a multiplication operator M_a on $l_2(I)$ so that $UAU^{-1} = M_a$ (thm. 6.3 [2])
- If there exists a unitary / orthogonal operator $U : H \to l_2(I)$ where $UAU^{-1} = M_a$ we say that A is *unitarily equivalent* to M_a and we say that U *diagonalizes* A (rem. 6.4 [2])
- The *Fourier transform* $F : L_2([-\pi,\pi]) \to l_2(\mathbb{Z})$ defined by $F(f) = (c_n(f))_{n\in\mathbb{Z}}$ where $c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$ for $f \in L_2([\pi,\pi])$ is a unitary operator (thm. 6.5 [2])

- A diagonalizable operator $U \in \mathcal{B}(H)$ is unitary $\Leftrightarrow$ it's eigenvalues are on the unit circle (i.e. have numeracal values 1) (thm. 6.6 [2])

8.1 Fourier Transform on $\mathbb{R}$

- The unitary operator the *Fourier transform* on $L_2(\mathbb{R}^n)$: $\mathcal{F} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ (lots of stuff on p. 6.8 - 6.9) (p. 6.8 [2])

- The *Fourier transformed* function $\widehat{f}$ of f: $\widehat{f}(k) \doteq \int_{\mathbb{R}} f(x)e^{-ikx}dx, k \in \mathbb{R}$ (def. 6.10 [2])

- $\forall f \in L_1(\mathbb{R}) : \widehat{f}$ is a continuous and limited function on $\mathbb{R}$ and $||\widehat{f}||_u \leq ||f||_1$ (thm. 6.11 [2])

- $\widehat{f}(k) \to 0$ for $|k| \to \infty$ (rem. 6.12 [2])

- $\forall f \in \mathcal{C}_0^\infty(\mathbb{R})$ the following holds:

  - $\widehat{f} \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$
  - *Fourier's formula of inversion*: $f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(k)e^{ikx}dk, x \in \mathbb{R}$
  - *Parseval equation*: $\int_{\mathbb{R}} |f(x)|^2dx = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(k)|^2dk$

  (thm. 6.13 [2])

- A function f on $\mathbb{R}$ is called a *Schwartz function* $(f \in \mathcal{S}(\mathbb{R})) \doteq f \in C^\infty(\mathbb{R})$ and $\forall n, N \in \mathbb{N} \cup \{0\} : \exists C_{n,N} \geq 0 : |f^{(n)}(x)| \leq \frac{C_{n,N}}{(1+|x|)^N}$ for $x \in \mathbb{R}$ where $f^{(n)}$ is the nth derivarive of f (def. 6.14 [2])

- If $f \in \mathcal{S}(\mathbb{R})$ we can conclude the following:

  1. $\forall x \in \mathbb{R} : |x^N f^{(n)}(x)| \leq C_{n,N}$ - i.e. f and all it's derivatives goes towards 0 faster than any power function for $|x| \to +\infty$
  2. $f \in \mathcal{S}(\mathbb{R}) \Rightarrow x^N f^{(n)}(x) \in \mathcal{S}(\mathbb{R})$
  3. $f \in \mathcal{S}(\mathbb{R}) \Rightarrow f \in \mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_2(\mathbb{R})$

  (rem. 6.15 [2])

- $\forall f \in \mathcal{S}(\mathbb{R}) : \forall n \in \mathbb{N} : \widehat{f} \in \mathcal{S}(\mathbb{R}) \wedge \widehat{f^{(n)}}(k) = (ik)^n\widehat{f}(k) \wedge \widehat{x^n f}(k) = (i\frac{d}{dk})^n\widehat{f}(k)$ (where $x^n f$ is $x \to x^n f(x)$ (lemma 6.16 [2])

- There exists a unique unitary operator $\mathcal{F} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$, the *Fourier transform* on $L_2(\mathbb{R}^n)$ where $\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}}\widehat{f}$ for $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R})$ bijectively into $\mathcal{S}(\mathbb{R})$ (thm. 6.17 [2])

- $\check{f}$ is called the *co-Fourier transformed* of f and is given by: $\check{f}(k) \doteq \int_{\mathbb{R}} f(x)e^{ikx}dx$ for $f \in L_1(\mathbb{R})$. We also have: $\check{f}(k) = \widehat{f}(-k)$ (p. 6.16 [2])

- If $\overline{\mathcal{F}} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is the *co-Fourier transform* we have: $f = \overline{\mathcal{F}}(\mathcal{F}(f))$ (p. 6.16 [2])

8.2 Fourier Transform on $\mathbb{R}^n$

- For $f \in L_1(\mathbb{R}^n)$ we define the Fourier transformed function $\widehat{f}$ of f by $\widehat{f}(k) = \int_{\mathbb{R}^n} f(x)e^{-ik\cdot x}dx$ for $k \in \mathbb{R}^n$ (where we use the dot product $k \cdot x = k_1x_1 + \cdots + k_nx_n$ for $k = (k_1, \ldots, k_n)$ and $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$) (def. 6.18 [2])
- A *multi index* is a set $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_1, \ldots, \alpha_n \in \mathbb{N} \cup \{0\}$ (p. 6.17 [2])
- For $x = (x_1, \ldots, x_n)$ and a multi index α we set $x^\alpha = x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}$ and correspondingly $\partial^\alpha = \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdot \cdots \cdot \partial_{x_n}^{\alpha_n}$. So ∂^α is a differential operator which can be used on e.g. C^∞ functions defined on $\mathbb{R}^n$ (p. 6.17 [2])
- $f : \mathbb{R}^n \to \mathbb{C}$ is called a *Schwartz function* $\underset{def}{=}$ $f \in C^\infty(\mathbb{R}^n)$ and for all multi indices α and all $N \in \mathbb{N} \cup \{0\}$ there exists a constant $C_{\alpha,N}$ such that $|\partial^\alpha f(x)| \leq \frac{C_{\alpha,N}}{(1+||x||)^N}$ for $x \in \mathbb{R}^n$ (def. 6.19 [2])
- $f \in \mathcal{S}(\mathbb{R}^n) \Leftrightarrow f \in C^\infty(\mathbb{R}^n)$ and $|x^\alpha \partial^\beta f(x)|$ is limited on $\mathbb{R}^n$ for all pairs of muti indices α, β (p. 6.17 [2])
- $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow x^\alpha \partial^\beta f(x) \in \mathcal{S}(\mathbb{R}^n)$ (p. 6.17 [1])
- $C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ (p. 6.17 [2])
- There exists a unique unitary operator $\mathcal{F}$ on $L_2(\mathbb{R}^n)$ such that $\mathcal{F}(f) = \frac{1}{(2\pi)^{n/2}}\widehat{f}$ for $f \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$. The following holds for $\mathcal{F}$:

  1. Maps $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$
  2. $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \mathcal{F}(\partial^\alpha f)(k) = (ik)^\alpha(\mathcal{F}(f))(k), k \in \mathbb{R}^n$ for any α
  3. $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \mathcal{F}((-ix)^\alpha f)(k) = \partial^\alpha(\mathcal{F}(f))(k), k \in \mathbb{R}^n$ for any α
  4. $f \in L_2(\mathbb{R}^n) \Rightarrow (\mathcal{F}^*(f))(k) = (\mathcal{F}(f))(-k), k \in \mathbb{R}^n$

  (thm. 6.20 [1])
- The *translation operator* on $L_2(\mathbb{R}^n)$ for $a \in \mathbb{R}^n$ $\underset{def}{=}$ $U_a : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ given by $(U_af)(x) = f(x+a), x \in \mathbb{R}^n$. U_a is isometric, bijective, limited, unitary (ex. 6.21 [1])
- The *multiplication operator* on $L_2(\mathbb{R}^n)$ for $k \mapsto e^{ik\cdot a}$ $\underset{def}{=}$ $M_{e^{ik\cdot a}} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ given by $(U_af)(x) = f(x+a), x \in \mathbb{R}^n$. $M_{e^{ik\cdot a}}$ is limited, unitary (ex. 6.21 [1])
- $\forall f \in L_2(\mathbb{R}^n) : \mathcal{F}(U_af) = M_{e^{ik\cdot a}}(\mathcal{F}(f))$, in other words: $\mathcal{F}U_a\mathcal{F}^{-1} = M_{e^{ik\cdot a}}$ (ex. 6.21 [1])
- The *Laplacian operator* on $\mathbb{R}^n$ $\underset{def}{=}$ $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ (ex. 6.22 [1])
- Let $M_{\omega^{-1}}$ be the multiplication operator for ω^{-1} (i.e. $1/\omega$). For $m^2 > 0$ and some given function f on $\mathbb{R}^n$ the solution ($u \in \mathcal{S}(\mathbb{R}^n)$) for the partial differential equation $(-\Delta + m^2I)u = f$ is $u = \mathcal{F}^{-1}M_{\omega^{-1}}\mathcal{F}f$. The operator $(-\Delta + m^2I)$ maps $\mathcal{S}(\mathbb{R}^n)$ bijectively onto itself. The inverse operator is $(-\Delta + m^2I)^{-1} = \mathcal{F}^{-1}M_{\omega^{-1}}\mathcal{F}$. The inverse is a limited self adjugated (?) operator. The forward operator is unlimited and is not defined everywhere (see p. 6.20 in [2]) for more info) (ex. 6.22 [2])

8.3 Misc Fourier Transform

- Let H be a Hilbert space over $\mathbb{R}$ and let O be a diagonalizable orthogonal operator on H. Let P be the orthogonal projection on the eigenspace $E_{-1}(O)$ then the orthogonal projection on the eigenspace $E_1(O)$ is $1 - P$ and $O = 1 - 2P$. Conversely any operator given by $O = 1 - 2P$ (where P is any orthogonal projection) is orthogonal and diagonalizable (ex. 6.7 [2])

- Let $H = L_2([-\pi, \pi])$ and consider the *translation operator* $T_\alpha : H \to H$ for $\alpha \in \mathbb{R}$ given by $(T_\alpha f)(\theta) = f(\alpha+\theta)$ for $\theta \in \mathbb{R}, f \in H$ then the following holds:

  1. $||T_\alpha f||^2 = ||f||^2$
  2. T_α is unitary
  3. $(e_n)_{n\in\mathbb{Z}}$ diagonalizes T_α and the eigenvalues are $e^{in\alpha}, n \in \mathbb{Z}$
  4. $FT_\alpha F^{-1}$ equals the multiplication operator on $l_2(\mathbb{Z})$ for the function $n \to e^{in\alpha}$ - i.e. $FT_\alpha F^{-1} = M_{e^{in\alpha}}$
  5. The Fourier transform on $L_2([-\pi, \pi])$ diagonalizes T_α

  (ex. 6.8 [2])

- The Fourier transform on $L_2([-\pi, \pi])$ in some sense diagonalizes the differential operator $D = -i\frac{d}{d\theta}$. For $f \in C^1_{per} : F(D(f)) = M_{id}F(f)$ where M_{id} is the identity function on $l_2(\mathbb{Z})$ i.e.: $M_{id} : (x_n)_{n\in\mathbb{Z}} \to (nx_n)_{n\in\mathbb{Z}}$ (more details in the text p. 6.7) (rem. 6.9 [2])

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