Geometric Surfaces Summary

Ánoq of the Sun, Hardcore Processing [*]

June 30, 2004

1 Vectors, Functions, Conventions etc.

- $U \underset{open}{\subseteq} \mathbb{R}^m$, the map $\overrightarrow{f} : U \to \mathbb{R}^n$ is *smooth* $\underset{def}{=}$ *each* of the n components of $\overrightarrow{f}$, $\overrightarrow{f}_i : U \to \mathbb{R}$ have *continuous partial derivatives of all orders* (i.e. they are C^∞).
  E.g. $m = 2, n = 3, f(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))$ and then:
  $\frac{\partial \overrightarrow{f}}{\partial u} = (\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u}, \frac{\partial f_3}{\partial u})$, $\frac{\partial \overrightarrow{f}}{\partial v} = (\frac{\partial f_1}{\partial v}, \frac{\partial f_2}{\partial v}, \frac{\partial f_3}{\partial v})$. (p. 66-67 [1])

- *Jacobian* $\mathcal{J}(f)$ of *smooth function* $f : U \to \mathbb{R}^n$ $\underset{def}{=}$

$$\mathcal{J}(f) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$$

  Where $U \underset{open}{\subseteq} \mathbb{R}^m$ and $(f_1, f_2, \dots, f_n) = f(u_1, \dots, u_m)$. (p. 93 [1])

1.0.1 Misc formulas:

- In a *quadratic equation*: $ax^2 + bx + c = 0$, the *sum* of the *roots* is $-b/a$ and the *product* of the roots is c/a. (p. 148 [1])

- $\forall a, b \in \mathbb{R} : |a - b| \geq ||a| - |b||$. (p. 168 [1])

- $\forall \overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d} \in \mathbb{R}^3 : (\overrightarrow{a} \times \overrightarrow{b}).(\overrightarrow{c} \times \overrightarrow{d}) = (\overrightarrow{a}.\overrightarrow{c})(\overrightarrow{b}.\overrightarrow{d}) - (\overrightarrow{a}.\overrightarrow{d})(\overrightarrow{b}.\overrightarrow{c})$. (p. 113 [1])

- $\frac{d}{d\tau} \int_{\mathbb{R}} f(\tau, t)dt = \int \frac{\partial f}{\partial \tau} dt$. (from proof 8.2 p. 192 [1])

- Let $f : U \to \mathbb{R}^n$ be a *smooth map* on $U \underset{open}{\subseteq} \mathbb{R}^n (n \geq 1)$. Assume that at *some point* $x_0 \in U$, the *Jacobian* matrix $\mathcal{J}(f)$ is *invertible.* Then there *exists* $V \underset{open}{\subseteq} \mathbb{R}^n$ and a *smooth map* $g : V \to \mathbb{R}^n$ such that:
  - $y_0 = f(x_0) \in V$
  - $g(y_0) = x_0$
  - $g(V) \subseteq U$
  - $g(V) \underset{open}{\subseteq} \mathbb{R}^n$

  - $\forall y \in V : f(g(y)) = y$

  In particular $g : V \to g(V)$ and $f : g(V) \to V$ are *inverse bijections.* (thm. 4.2, Inverse function theorem, [1])

1.0.2 Conventions:

- Abbreviations: $\frac{\partial \overrightarrow{f}}{\partial u} = \overrightarrow{f}_u, \frac{\partial \overrightarrow{f}}{\partial v} = \overrightarrow{f}_v$ and $\frac{\partial^2 \overrightarrow{f}}{\partial u^2} = \overrightarrow{f}_{uu}, \frac{\partial^2 \overrightarrow{f}}{\partial v^2} = \overrightarrow{f}_{vv}, \frac{\partial^2 \overrightarrow{f}}{\partial u \partial v} = \overrightarrow{f}_{uv}$ (p. 66-67 [1])
- From page 72 in [1], all *surfaces* are *smooth* and all surface *patches* are *regular smooth* patches - and unless otherwise noted they are also assumed to be *connected.*

2 Surface Representations

- $\mathcal{S} \subseteq \mathbb{R}^3$ is a surface $\underset{def}{=} \forall p \in S : \exists U \underset{open}{\subseteq} \mathbb{R}^2, W \underset{open}{\subseteq} \mathbb{R}^3 : p \in W \wedge (\mathcal{S} \cap W$ is homeomorphic to $U)$. The homeomorphisms $\overrightarrow{\sigma} : U \to S \cap W$ are called *surface pathces* or *parametrisations.* The collection of all these surface patches is called the *atlas* for $\mathcal{S}$. Thus every point of $\mathcal{S}$ lies in the image of at least one patch in the atlas of $\mathcal{S}$. (def. 4.1 [1])

2.0.3 Representations:

- *Parametrised patch* for $\mathcal{S} \subseteq \mathbb{R}^3$:
  Homeomorphism $\overrightarrow{\sigma} : U \to S$ where $U \subseteq \mathbb{R}^2$ (def. 4.1 [1])
- *Smooth function*: Surface defined by $f(x, y, z) = 0$ where f is a *smooth function* such that f_x, f_y, f_z do *not all vanish* at *any point* of $\mathcal{S}$. (exc. 4.16 [1])
- *Graph*: $f(x, y)$ *smooth function* $\Rightarrow$
  it's graph $\{(x, y, x) \in \mathbb{R}^3 \mid z = f(x, y)\}$ is a *smooth surface* with atlas consisting of the patch $\overrightarrow{\sigma}(u, v) = (u, v, f(u, v))$. (exc. 4.6 [1])

2.0.4 Conversion between representations:

- *Parametrized to parametrized*: Let $\overrightarrow{\sigma} : U \to \mathcal{S} \cap W$ and $\overrightarrow{\widetilde{\sigma}} : \widetilde{U} \to \mathcal{S} \cap \widetilde{W}$ be two patches for the surface S such that $a \in \mathcal{S} \cap W \cap \widetilde{W}$. Since $\overrightarrow{\sigma}$ and $\overrightarrow{\widetilde{\sigma}}$ are homeomorphisms we have that $V = \overrightarrow{\sigma}^{-1}(\mathcal{S} \cap W \cap \widetilde{W})$ and $\widetilde{V} = \overrightarrow{\widetilde{\sigma}}^{-1}(\mathcal{S} \cap W \cap \widetilde{W})$ are open sets $V \underset{open}{\subseteq} U$ and $\widetilde{V} \underset{open}{\subseteq} \widetilde{U}$ respectively. The composite homeomorphism $\overrightarrow{\phi} = \overrightarrow{\sigma}^{-1} \circ \overrightarrow{\widetilde{\sigma}} : \widetilde{V} \to V$ is called the *transition map* from $\overrightarrow{\sigma}$ to $\overrightarrow{\widetilde{\sigma}}$.
  We also have that $\forall(\widetilde{u}, \widetilde{v}) \in \widetilde{V} : \overrightarrow{\widetilde{\sigma}}(\widetilde{u}, \widetilde{v}) = \overrightarrow{\sigma}(\phi(\widetilde{u}, \widetilde{v}))$. (p. 64-65 [1])
- $\overrightarrow{\widetilde{\sigma}}$ is a *reparametrisation* of $\overrightarrow{\sigma}$ with *reparametrisation map* $\overrightarrow{\Phi} \underset{def}{=}$
  Let $U, \widetilde{U} \underset{open}{\subseteq} \mathbb{R}^2$ and let $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ be a *regular surface patch.* Let $\overrightarrow{\Phi} : \widetilde{U} \to U$ be a *bijective smooth map* with *smooth inverse map* $\overrightarrow{\Phi}^{-1} : U \to \widetilde{U}$. Then $\overrightarrow{\widetilde{\sigma}} = \overrightarrow{\sigma} \circ \overrightarrow{\Phi} : \widetilde{U} \to \mathbb{R}^3$ is a *regular surface patch.* (prop. 4.2 [1])

3 Some Well-known Named Surfaces

The implicit formulas denote the sets $\{(x, y, z) \in \mathbb{R}^3 \mid \ldots\}$. The patches are given as $\overrightarrow{\sigma}(u, v) = \ldots$ or $\overrightarrow{\sigma}(\theta, \phi) = \ldots$.

Name	Implicit	Patch / Parametrisation	Param Rg	Ref.
Plane		$\overrightarrow{a} + u\overrightarrow{p} + v\overrightarrow{q}$	$\mathbb{R} \times \mathbb{R}$	(ex. 4.1 [1])
Open disc				(exc. 4.1 [1])
Unit sphere S^2	$x^2 + y^2 + z^2 = 1$	$(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$	$]\frac{-\pi}{2}, \frac{\pi}{2}[\times]0, 2\pi[$	(ex. 4.2 [1])
- extra patch		$(-\cos\theta\cos\phi, -\sin\theta, -\cos\theta\sin\phi)$	$]\frac{-\pi}{2}, \frac{\pi}{2}[\times]0, 2\pi[$	
- 6 alt. patches		$(\pm\sqrt{1 - u^2 - v^2}, u, v)$ (4 others)	$u^2 + v^2 < 1$	(exc. 4.3 [1])
Double cone	$x^2 + y^2 = z^2$	$(u, v, \pm\sqrt{u^2 + v^2})$	$\mathbb{R}^3 \setminus \{(0, 0)\}$	(ex. 4.3 [1])
Circ. cylinder	$x^2 + y^2 = 1$			(exc. 4.2 [1])
Hyperboloid 1	$x^2 + y^2 - z^2 = 1$			(exc. 4.4 [1])

4 Local Surface Properties

4.0.5 For surfaces $\mathcal{S}$:

- *Tangent space* at point P of $\mathcal{S}$ $\underset{def}{=}$ the *set* of *tangent vectors* at P of *all curves* in $\mathcal{S}$ passing *through* P. (def. 4.4 [1])

4.0.6 For *parametrised* surfaces $\mathcal{S}$:

$\vec{\sigma} : U \to \mathbb{R}^3$.

- (*Standard*) *unit-normal* of $\vec{\sigma}$ at P $\underset{def}{=}$ $\vec{N}_{\vec{\sigma}} = \frac{\vec{\sigma}_u \times \vec{\sigma}_v}{\|\vec{\sigma}_u \times \vec{\sigma}_v\|}$ *evaluated* at the *point of* U *corresponding to* P. (p. 75 [1])
- *First fundamental form* $\underset{def}{=}$ $Edu^2 + 2Fdudv + Gdv^2$ (p. 98 [1])
  $E = \|\vec{\sigma}_u\|^2 = \vec{\sigma}_u.\vec{\sigma}_u$,
  $F = \vec{\sigma}_u.\vec{\sigma}_v$,
  $G = \|\vec{\sigma}_v\|^2 = \vec{\sigma}_v.\vec{\sigma}_v$. (p. 97-98 [1])
  First fundamental form will change when *patch is changed.* (exc. 5.4 [1])
- *Second fundamental form* of $\vec{\sigma}$ $\underset{def}{=}$ $Ldu^2 + 2Mdudv + Ndv^2$, where:
  $L = \vec{\sigma}_{uu}.\vec{N}$
  $M = \vec{\sigma}_{uv}.\vec{N}$
  $N = \vec{\sigma}_{vv}.\vec{N}$
  and $\vec{N}$ is the standard unit normal. (p. 124-125 [1])
- *Third fundamental form* of $\vec{\sigma}$ $\underset{def}{=}$ $\|\vec{N}_u\|^2du^2 + 2\vec{N}_u.\vec{N}_v dudv + \|\vec{N}_v\|^2dv^2$. (exc. 6.22 [1])
- $\mathcal{F}_I = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$. (p. 132 [1])
- $\mathcal{F}_{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$. (p. 132 [1])
- $\mathcal{F}_{III}$ = (matrix of 3. fundamental form) $= \mathcal{F}_{II}\mathcal{F}_I^{-1}\mathcal{F}_{II}$. (exc. 6.22 [1])
- The *principal curvatures* of $\vec{\sigma}$ are the *roots* of the equation:
  $det(\mathcal{F}_{II} - \kappa\mathcal{F}_I) = 0$ i.e. $\begin{Vmatrix} L-\kappa E & M-\kappa F \\ M-\kappa F & N-\kappa G \end{Vmatrix} = 0$.
  There are 2 *real roots* and thus there exists a non-zero 2×1 matrix T of real numbers such that: $(\mathcal{F}_{II} - \kappa\mathcal{F}_I)T = 0$. (def. 6.1 [1])
- A *principal vector* $\vec{t}$ corresponding to the *principal* curvature κ $\underset{def}{=}$ $\vec{t} = \xi\vec{\sigma}_u + \eta\vec{\sigma}_v$, where $T = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$ satisfies $(\mathcal{F}_{II} - \kappa\mathcal{F}_I)T = 0$. (def. 6.2 [1])
- Let $\vec{N}$ be the *standard unit normal* of $\vec{\sigma}$. Then:
  $\vec{N}_u = a\vec{\sigma}_u + b\vec{\sigma}_v$, $\vec{N}_v = c\vec{\sigma}_u + d\vec{\sigma}_v$ where
  $\mathcal{W} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = -\mathcal{F}_I^{-1}\mathcal{F}_{II}$.
  $\mathcal{W}$ is the *Weingarten matrix* of $\vec{\sigma}$. (prop. 6.4 [1])

- The *gaussian curvature* K of $\overrightarrow{\sigma} \underset{def}{=} K = \kappa_1 \kappa_2$. (def. 7.1 [1])
- The *mean curvature* H of $\overrightarrow{\sigma} \underset{def}{=} H = \frac{1}{2}(\kappa_1 + \kappa_2)$. (def. 7.1 [1])
  (some authors omit the factor $\frac{1}{2}$)
- The *Gauss map* $\mathcal{G} \underset{def}{=} \mathcal{G} : \mathcal{S} \to \mathcal{S}^2$ where $\mathcal{S}$ is the image of $\overrightarrow{\sigma}$ and $\mathcal{G}$ sends the point $\overrightarrow{\sigma}(u, v)$ of $\mathcal{S}$ to $\overrightarrow{N}(u, v)$ of $\mathcal{S}^2$.
  $\mathcal{G}$ is defined for any *orientable surface* $\mathcal{S}$. (p. 166 [1])
- The *Cristoffel symbols* are defined by:
  - $\overrightarrow{\sigma}_{uu} = \Gamma^1_{11} \overrightarrow{\sigma}_u + \Gamma^2_{11} \overrightarrow{\sigma}_v + L\overrightarrow{N}$
  - $\overrightarrow{\sigma}_{uv} = \Gamma^1_{12} \overrightarrow{\sigma}_u + \Gamma^2_{12} \overrightarrow{\sigma}_v + M\overrightarrow{N}$
  - $\overrightarrow{\sigma}_{vv} = \Gamma^1_{22} \overrightarrow{\sigma}_u + \Gamma^2_{22} \overrightarrow{\sigma}_v + N\overrightarrow{N}$

  where

  - $\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - FF^2)}$, $\Gamma^2_{11} = \frac{EF_u - 2EE_v + FE_u}{2(EG - FF^2)}$
  - $\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - FF^2)}$, $\Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - FF^2)}$
  - $\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_u}{2(EG - FF^2)}$, $\Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - FF^2)}$

  (prop. 10.3 [1])

5 Local Surface Theorems

5.0.7 For surfaces $\mathcal{S}$:

- Let $\overrightarrow{\sigma}$ be a patch *containing* point P of $\mathcal{S}$. Let (u, v) be coordinates of U.
  The *tangent space* (a.k.a. the *tangent plane*) to $\mathcal{S}$ at P is the *vector subspace* of $\mathbb{R}^3$ *spanned* by the vectors $\overrightarrow{\sigma}_u$ and $\overrightarrow{\sigma}_u$ (evaluated at point $(u_0, v_0) \in U$ such that $\overrightarrow{\sigma}(u_0, v_0) = P$). (prop. 4.4 [1])

- The *set of linear combinations* of $\overrightarrow{\sigma}_u$ and $\overrightarrow{\sigma}_v$ are *unchanged* when $\overrightarrow{\sigma}$ is *reparametrised.* (exc. 4.15 [1])

- *Tangent space / tangent plane* is *independent* of the *choice of patch* containing P. (p. 75 [1])

- For a *surface defined by a smooth function* $f(x, y, z) = 0$, the vector $\overrightarrow{\nabla f} = (f_x, f_y, f_z)$ is *perpendicular* to the *tangent plane* at *every point* of $\mathcal{S}$ and thus $\mathcal{S}$ is *orientable.* (exc. 4.16 [1])

- Let $F : \mathcal{S} \to \mathbb{R}$ be a *smooth function* then:
  - $\forall P \in \mathcal{S}$: there exists a *unique vector* $\overrightarrow{\nabla_{\mathcal{S}} F}$ in the *tangent plane* at P such that $(\overrightarrow{\nabla_{\mathcal{S}} F}).\overrightarrow{\gamma}'(0) = \frac{d}{dt}|_{t=0} F(\overrightarrow{\gamma}(t))$ for all curves $\overrightarrow{\gamma}$ in $\mathcal{S}$ with $\overrightarrow{\gamma}(0) = P$
  - If F has a *local maximum* or *minimum* at P then $\overrightarrow{\nabla_{\mathcal{S}} F} = 0$
  - If $\mathcal{S}$ is defined by $f(x, y, z) = 0$ where f is *smooth* and f_x, f_y, f_z do *not all vanish* at any point of $\mathcal{S}$ then $\overrightarrow{\nabla_{\mathcal{S}} F}$ is the *perdendicular projection* of $\overrightarrow{\nabla F}$ onto the *tangent plane* to $\mathcal{S}$ and then if F has *local minimum* or *maximum* at P then $\overrightarrow{\nabla F} = \lambda \overrightarrow{\nabla f}$ for some scalar λ

  (exc. 4.17, Lagrange's Method of Undetermined Multipliers [1])

- Applying a *rigid motion* to a *surface* does *not change* it's *first fundamental form.* (exc. 5.3 [1])

- Applying a *rigid motion* to a *surface* does *not change* it's *second fundamental form.* (exc. 6.4 [1])

- Let $\overrightarrow{\tilde{\sigma}}(\tilde{u}, \tilde{v})$ be a *reparametrisation* of $\overrightarrow{\sigma}(u, v)$ and their first fundamental forms be $\widetilde{E}, \widetilde{F}, \widetilde{G}$ and E, F, G respectively.
  Let $\mathcal{J} = \begin{bmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{bmatrix}$ be the Jacobian matrix of the reparametrisation map $(\tilde{u}, \tilde{v}) \mapsto (u, v)$ and let $\mathcal{J}^t$ be the transpose of $\mathcal{J}$. Then:
  $\begin{bmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{bmatrix} = \mathcal{J}^t \begin{bmatrix} E & F \\ F & G \end{bmatrix} \mathcal{J}$. (exc. 5.4 [1])

- A *diffeomorphism* $f : \mathcal{S}_1 \to \mathcal{S}_2$ is an *isometry* $\Leftrightarrow$ for any surface patch $\overrightarrow{\sigma}_1$ in a given atlas of $\mathcal{S}_1$, the patches $\overrightarrow{\sigma}_1$ and $f \circ \overrightarrow{\sigma}_1$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively have the *same first fundamental form.* (thm. 5.1 [1])

- The *principal curvatures* κ_1, κ_2 at point P of $\mathcal{S}$ are the *maximum* and *minimum* values of the *normal curvature* of all curves on $\mathcal{S}$ that pass through P. The *principal vectors* at point P of $\mathcal{S}$ are the *tangent vectors* of the curves giving these minimum and maximum values. (cor. 6.2 [1])
- Near *point* P of $\mathcal{S}$, $\mathcal{S}$ *conincides* with the *quadric surface*:
  $z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2)$. Classification of shapes (and quadrics) at P:
  - κ_1, κ_2 *both* > 0 or *both* < 0: *Elliptic point* (elliptic paraboloid)
  - $\kappa_1 = \kappa_2$: *Umbilic (and elliptic) point* (sphere) (p. 133 [1])
  - $\kappa_1, \kappa_2 = 0$: *planar (and umbilic) point* (plane) - we must use higher-order derivatives to determine the shape
  - κ_1, κ_2 *non-zero* and *opposite sign*: *hyperbolic point* (hyperbolic paraboloid)
  - κ_1, κ_2 *one zero* and *one non-zero*: *parabolic point* (parabolic cylinder)

  The classification is *independent* of $\vec{\sigma}$. (p. 142 [1])
- The *gaussian curvature* at a point P of $\mathcal{S}$:
  - $K > 0$ at $P : P$ is an *elliptic point*
  - $K < 0$ at $P : P$ is an *hyperbolic point*
  - $K = 0$ at $P : P$ is either *parabolic* or *planar*

  (p. 164 [1])

5.0.8 For *parametrised* surfaces $\mathcal{S}$:

$\vec{\sigma} : U \to \mathbb{R}^3$.

- $\overrightarrow{N}_{\vec{\sigma}}$ is *perpendicular* to *every linear combination* of $\vec{\sigma}_u$ and $\vec{\sigma}_v$. (p. 75 [1])
- If $\vec{\sigma}$ and $\vec{\tilde{\sigma}}$ are *2 patches for a surface*, then:
  $\vec{\tilde{\sigma}}_{\tilde{u}} \times \vec{\tilde{\sigma}}_{\tilde{v}} = \det(\mathcal{J}(\Phi))\vec{\sigma}_u \times \vec{\sigma}_v$ and $\overrightarrow{N}_{\vec{\tilde{\sigma}}} = \pm\overrightarrow{N}_{\vec{\sigma}}$ where the *sign* is the *sign of* $\det(\mathcal{J}(\Phi))$. (p. 75 [1])
- $\mathcal{F}_{III} - 2H\mathcal{F}_{II} + K\mathcal{F}_I = 0$ on $\vec{\sigma}$. (exc. 7.9 [1])
- Let κ_1 and κ_2 be the *principal curvatures* at point P of $\vec{\sigma}$, then:
  - κ_1 and κ_2 are *real numbers*
  - There exists a non-zero 2×1 matrix T of real numbers such that: $(\mathcal{F}_{II} - \kappa\mathcal{F}_I)T = 0$ (def. 6.1 [1])
  - $\kappa_1 = \kappa_2 (= \kappa) \Rightarrow \mathcal{F}_{II} = \kappa\mathcal{F}_I$, and hence *every tangent* vector to $\vec{\sigma}$ at P is a *principal vector*
  - $\kappa_1 \neq \kappa_2 \Rightarrow$ *any two* non-zero *principal vectors* $\vec{t}_1$, $\vec{t}_2$ korresponding to κ_1, κ_2 respectively are *perpendicular.*

  (prop. 6.3 [1])
- κ_1 and κ_1 either *stay the same* or *both change sign* when $\vec{\sigma}$ is *reparametrised*, according to whether $\overrightarrow{N}$ stays the same or changes sign. (exc. 6.17 [1])

- The *principal vectors* are *unchanged* by *reparametrisation.* (exc. 6.17 [1])
- *Gaussian curvature stays the same* if $\overrightarrow{\sigma}$ is *reparametrised.* (p. 148 [1])
- *Gaussian curvature stays the same* when applying a *rigid motion.* (exc. 7.6 [1])
- *Gaussian curvature* is *multiplied by* a^{-2} when applying the *dilation* $(x, y, z) \mapsto a(x, y, z)$, where a is a non-zero constant. (exc. 7.6 [1])
- *Mean curvature* either *stays the same* or *changes sign* when $\overrightarrow{\sigma}$ is *reparametrised.* (p. 148 [1])
- *Mean curvature stays the same* when applying a *rigid motion.* (exc. 7.6 [1])
- *Mean curvature* is *multiplied by* a^{-1} when applying the *dilation* $(x, y, z) \mapsto a(x, y, z)$, where a is a non-zero constant. (exc. 7.6 [1])
- *Gaussian* and *mean curvatures* of $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ are *smooth functions* on U. (exc. 7.7 [1])
- $K = \frac{LN-M^2}{EG-F^2}$. (prop. 7.1 [1])
- $H = \frac{LG-2MF+NE}{2(EG-F^2)}$. (prop. 7.1 [1])
- *Principal curvatures* $\kappa_1, \kappa_2 = H \pm \sqrt{H^2 - K}$ respectively. (prop. 7.1 [1])
- *Principal curvatures* of $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ are *smooth functions* on any $W \underset{open}{\subseteq} U$ where W has *no umbilics.* (exc. 7.7 [1])
- Let $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ be a surface, let $(u_0, v_0) \in U$, and let $\delta > 0$ be such that the *closed disc* $R_\delta = \{(u, v) \in \mathbb{R}^2 \mid (u - u_0)^2 + (v - v_0)^2 \leq \delta^2\}$ with centre (u_0, v_0) and radius δ is *contained* in U (such δ exists because U open). Then:
  $\lim_{\delta \to 0} \frac{\mathcal{A}_{\overrightarrow{N}}(R_\delta)}{\mathcal{A}_{\overrightarrow{\sigma}}(R_\delta)} (= \lim_{\delta \to 0} \frac{\int\int_{R_\delta} \|\overrightarrow{N}_u \times \overrightarrow{N}_v\| dudv}{\int\int_{R_\delta} \|\overrightarrow{\sigma}_u \times \overrightarrow{\sigma}_v\| dudv}) = |K|$
  where K is the gaussian curvature of $\overrightarrow{\sigma}$ at $\overrightarrow{\sigma}(u_0, v_0)$. (thm. 7.1 [1])
  The *sign* of K can be recovered by defining the *signed area* of $\overrightarrow{N}(R)$ to be $\pm\mathcal{A}_{\overrightarrow{N}}(R)$ according to whether $\overrightarrow{N}_u \times \overrightarrow{N}_v$ points in the same or the opposite direction of $\overrightarrow{N}$. Then K becomes the limit of the ratio $\frac{signed\ area\ of\ \overrightarrow{N}(R)}{area\ of\ \overrightarrow{\sigma}(R)}$. (p. 168 [1])
- The following holds when using the Christoffel symbols:
  1. $L_v - M_u = L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2$
  2. $M_v - N_u = L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N\Gamma_{12}^2$

  (prop. 10.4, Codazzi-Mainardi Equations [1])
- If $F = 0$ and $M = 0$ of the *first* and *second fundamental forms* of a surface then:
  $L_v = \frac{1}{2}E_v(\frac{L}{E} + \frac{N}{G})$, $N_u = \frac{1}{2}G_u(\frac{L}{E} + \frac{N}{G})$.
  Furthermore the *princial curvatures* $\kappa_1 = L/E, \kappa_2 = N/G$ satisfy:
  $(\kappa_1)_v = \frac{E_v}{2E}(\kappa_2 - \kappa_1)$, $(\kappa_2)_u = \frac{G_u}{2G}(\kappa_1 - \kappa_2)$. (exc. 10.11 [1])

6 Global Pointwise Local Surface Properties

6.0.9 For parametrised surface patches:

$\overrightarrow{\sigma}(u,v) : U \to \mathbb{R}^3$:

- $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ *regular* $\underset{def}{=}$ $\overrightarrow{\sigma}$ *smooth* and vectors $\overrightarrow{\partial u}$ and $\overrightarrow{\partial v}$ are *linearly independent at all points* $(u,v) \in U$.
  Equivalently: $\overrightarrow{\sigma}$ *smooth* and $\forall p \in U : \overrightarrow{\partial u} \times \overrightarrow{\partial v}$ *non-zero.* (def. 4.2 [1])

6.0.10 For surfaces:

- *Smooth surface* $\underset{def}{=}$
  a surface whose *atlas* consists of *regular* surface patches. (def. 4.3 [1])

- *Maximal atlas* $\underset{def}{=}$ atlas consisting of *all regular* surface patches.
  The maximal atlas is *independent of* any *arbitrary choices.*
  Consists of patches $\overrightarrow{\sigma} : U \to \mathcal{S} \cap W$, for $U \underset{open}{\subseteq} \mathbb{R}^2, W \underset{open}{\subseteq} \mathbb{R}^3$ (p. 65 [1])

- The surface patches of the maximal atlas of $\mathcal{S}$ are called the *allowable* surface pathces for $\mathcal{S}$. (p. 68 [1])

- An *orientable* surface $\underset{def}{=}$ surface with the property that if Φ is the *transition map between any two surface patches* in the *atlas*, then $\det(\mathcal{J}(\Phi)) > 0$ where Φ is defined. (def. 4.5 [1])

- A *diffeomorphism* $f : \mathcal{S}_1 \to \mathcal{S}_2$ is an *isometry* $\underset{def}{=}$
  f takes *curves* in $\mathcal{S}_1$ to *curves of the same length* in $\mathcal{S}_2$.
  If there *exists* an *isometry* $f : \mathcal{S}_1 \to \mathcal{S}_2$, $\mathcal{S}_1$ and $\mathcal{S}_2$ are *isometric.* (def. 5.1 [1])

- A *diffeomorphism* $f : \mathcal{S}_1 \to \mathcal{S}_2$ is *conformal* $\underset{def}{=}$
  When f takes *curves* $\overrightarrow{\gamma}_1$ and $\overrightarrow{\tilde{\gamma}}_1$ in $\mathcal{S}_1$ to *curves* $\overrightarrow{\gamma}_2$ and $\overrightarrow{\tilde{\gamma}}_2$ in $\mathcal{S}_2$, the *angle of intersection* of $\overrightarrow{\gamma}_1$ and $\overrightarrow{\tilde{\gamma}}_1$ is *equal* to that of $\overrightarrow{\gamma}_2$ and $\overrightarrow{\tilde{\gamma}}_2$. (def. 5.2 [1])

- $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ is a *conformal parametrisation* or a conformal surface patch of $\mathcal{S}$ $\underset{def}{=}$ $\overrightarrow{\sigma}$ between surfaces U (the plane) and $\mathcal{S}$ is *conformal.* (p. 108 [1])

- $f_u = g_v$ and $f_v = -g_u$ are called the *Cauchy-Riemann equations* and are the condition for the map $u + iv \mapsto f(u,v) + ig(u,v)$ to be *holomorphic.* (exc. 5.14 [1])

- $f_u = -g_v$ and $f_v = g_u$ says that $u + iv \mapsto f(u,v) + ig(u,v)$ is *anti-holomorphic* - i.e. that the *complex conjugate* of the map is *holomorphic.* (exc. 5.14 [1])

- *Approximate surface area* at $(u_0, v_0) \in U$ for small Δu and Δv is:
  $\|\overrightarrow{\sigma}_u \Delta_u \times \overrightarrow{\sigma}_v \Delta_v\| = \|\overrightarrow{\sigma}_u \times \overrightarrow{\sigma}_v\| \Delta u \Delta v$. (p. 112-113 [1])

- *Surface area* of the part $\overrightarrow{\sigma}(R)$ of surface patch $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ where $R \subseteq U$ $\underset{def}{=}$ $\mathcal{A}_{\overrightarrow{\sigma}}(R) = \int\int_R \|\overrightarrow{\sigma}_u \times \overrightarrow{\sigma}_v\| dudv$. (def. 5.3 [1])
  $\mathcal{A}$ may be infinite, but finite if R is bounded entirely within U.

- A *diffeomorphism* $f : \mathcal{S}_1 \to \mathcal{S}_2$ is *equiareal* $\underset{def}{=}$
  f takes *any region* in $\mathcal{S}_1$ to *a region of the same area* in $\mathcal{S}_2$. (def. 5.4 [1])

- The *anitpodal points* on a *sphere* $\underset{def}{=}$ points directly on the opposite side. (p. 119 [1])

- *Flat surface* $\underset{def}{=}$ surface where $K = 0$ everywhere. (p. 155 [1])

- *Parallel surface* $\overrightarrow{\sigma}^\lambda$ of $\overrightarrow{\sigma}$ $\underset{def}{=}$ $\overrightarrow{\sigma}^\lambda = \overrightarrow{\sigma} + \lambda\overrightarrow{N}$, $\lambda \in \mathbb{R}$. (def. 7.2 [1])

- T_g $\underset{def}{=}$ the surface obtained by *joining g tori*. g is the *genus* of the surface and T_0 is the sphere, T_1 is the torus. (p. 258 [1])

- A *Quadric* $\underset{def}{=}$ a subset of $\mathbb{R}^3$ defined by an equation of the form $(\overrightarrow{r}A).\overrightarrow{r} + \overrightarrow{b}.\overrightarrow{r} + c = 0$ where $\overrightarrow{r} = (x, y, z)$ and A, $\overrightarrow{b}$, c *constants* and A *symmetric* 3x3 matrix, $\overrightarrow{b} \in \mathbb{R}^3, c \in \mathbb{R}$. Not always a surface (alternatives: point, line, intersecting planes). (def. 4.6 [1])

- A *doubly ruled surface* $\underset{def}{=}$ the union of each of two families of straight lines such that no two lines of the same family intersect, but every line of the first family intersects every line of the second family with at most a finite number of exceptions. (exc. 4.27 [1])

- *Triply orthogonal system* $\underset{def}{=}$ 3 families of surfaces, each family depending on a single parameter, with the property that, if P is a point on one surface of each family, the tangent planes of those surfaces are mutually perpendicular. (p. 90 [1])

6.0.11 For *vector fields* on surfaces:

- A *smooth tangent vector field* $\overrightarrow{V}$ on $\mathcal{S}$ $\underset{def}{=}$
  if $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ is a patch of $\mathcal{S}$ then $\overrightarrow{V} = \alpha(u, v)\overrightarrow{\sigma}_u + \beta(u, v)\overrightarrow{\sigma}_v$ where $\alpha, \beta : U \to \mathbb{R}$. The smoothness is independent of the choice of $\overrightarrow{\sigma}$. (p. 269 [1])

- P is a *stationary point* of a *smooth tangent vector field* $\overrightarrow{V}$ $\underset{def}{=}$
  $\overrightarrow{V} = \overrightarrow{0}$ at P of $\mathcal{S}$. (def. 11.5 [1])

- An *integral curve* $\overrightarrow{\gamma}$ of the *smooth tangent vector field* $\overrightarrow{V}$ through P $\underset{def}{=}$ the *unique curve* with $\overrightarrow{\gamma}' = \overrightarrow{V}$ and $\overrightarrow{\gamma}(0) = P$. (p. 269 [1])

- The *multiplicity* of the *stationary point* P of $\overrightarrow{V}$ on $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ $\underset{def}{=}$
  $\mu(P) = \frac{1}{2\pi}\int_0^{l(\overrightarrow{\gamma})} \frac{d\psi}{ds} ds$ where P is the *only stationary* point in U and $\overrightarrow{\gamma}(s)$ is *any positively-oriented unit-speed simple closed curve* in $\overrightarrow{\sigma}(U)$ with P *in it's interior* and *length* $l(\overrightarrow{\gamma})$. ψ is the *angle* between $\overrightarrow{V}$ and *some nowhere vanising smooth tangent vector field* (the reference tangent vector field).
  μ is an *integer* which is *independent* of the choices of *reference vector field* and *simple closed curve* $\overrightarrow{\gamma}$. (def. 11.6, p. 269 [1])

6.0.12 Critical Points

Let $\mathcal{S}$ be a *smooth* surface:

- (u_0, v_0) is a *critical point* of *smooth function* $f(u, v) \underset{def}{=} \partial f/\partial u$ and $\partial f/\partial v$ both vanish at (u_0, v_0). Equivalently the gradient of f, $\overrightarrow{\nabla f} = (\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})$ should vanish at (u_0, v_0).
  (Note: Here it is a property of functions, not surfaces) (p. 275 [1])

- Let $F : \mathcal{S} \to \mathbb{R}$ be a *smooth function* on $\mathcal{S}$. Point P of $\mathcal{S}$ a *critical point* of $F \underset{def}{=}$ there *exists* a surface patch $\overrightarrow{\sigma}$ of $\mathcal{S}$ with $P = \overrightarrow{\sigma}(u_0, v_0)$ such that $f = F \circ \overrightarrow{\sigma}$ has a critical point at (u_0, v_0). This definition is independent of the choice of $\overrightarrow{\sigma}$. (def. 11.7 [1])

- A *critical point* P of *smooth function* F on $\mathcal{S}$ is *non-degenerate* $\underset{def}{=}$ whenever $\overrightarrow{\sigma}(u, v)$ is a patch of $\mathcal{S}$ with $P = \overrightarrow{\sigma}(u_0, v_0)$ the matrix $\mathcal{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial u^2} & \frac{\partial^2 f}{\partial u \partial v} \\ \frac{\partial^2 f}{\partial u \partial v} & \frac{\partial^2 f}{\partial v^2 \partial v} \end{bmatrix}$ is *invertible*, where $f = F \circ \overrightarrow{\sigma}$ and the derivatives are evaluated at (u_0, v_0). In this case, the point P is called a *local maximum*, a *saddle point* or a *local minimum* if $\mathcal{H}$ has 2, 1 or 0 eigenvalues, respectively. This definition is independent of the choice of $\overrightarrow{\sigma}$. $\mathcal{H}$ is real and symmetric, so it always has two real eigenvalues (not necessarily distinct). (def. 11.8 [1])

6.0.13 For smooth surfaces $\mathcal{S}$:

- Let $\mathcal{S}_1, \mathcal{S}_2$ be smooth surfaces, assumed to be covered by the single surface patches $\overrightarrow{\sigma}_1 : U_1 \to \mathbb{R}^3$ and $\overrightarrow{\sigma}_2 : U_2 \to \mathbb{R}^3$.
  Then *Smooth map* $f : \mathcal{S}_1 \to \mathcal{S}_2 \underset{def}{=} \overrightarrow{\sigma}_1 \circ f \circ \overrightarrow{\sigma}_2$ *smooth.* (p. 69 [1])

- For $\mathcal{S}$ covered by a single surface patch $\overrightarrow{\sigma}$, a *smooth function* $f : \mathcal{S} \to \mathbb{R} \underset{def}{=} f$ *smooth* $\Leftrightarrow f \circ \overrightarrow{\sigma}$ *smooth.* (exc. 4.11 [1])

- f *diffeomorphism* $\underset{def}{=} f : \mathcal{S}_1 \to \mathcal{S}_2$ *smooth map* which is *bijective* and whose *inverse map* $f^{-1} : \mathcal{S}_2 \to \mathcal{S}_1$ is *smooth.*
  $\mathcal{S}_1$ and $\mathcal{S}_2$ are then said to be *diffeomorphic.* (p. 70 [1])

- Let $f : \mathcal{S}_1 \to \mathcal{S}_2$ be a *diffeomorphism.*
  $\overrightarrow{\sigma}_1$ *allowable surface patch* on $\mathcal{S}_1 \Rightarrow f \circ \overrightarrow{\sigma}_1$ *allowable surface patch* on $\mathcal{S}_2$. (prop. 4.3 [1])

7 Global Pointwise Local Surface Theorems

7.0.14 For surfaces $\mathcal{S}$:

- *Any surface* is a *disjoint union* of *connected surfaces.* (p. 72 [1])
- An *orientable* surface $\mathcal{S}$ has a *canonical choice* of *unit-normal* at each point obtained be taking the *standard unit-normal* of *each surface patch* in the *atlas* of $\mathcal{S}$. (prop. 4.5 [1])
- $\mathcal{S}$ has a *unit normal* $\overrightarrow{N}$ defined at *each point* $P \in \mathcal{S}$ and *depending smoothly* on $P \Rightarrow \mathcal{S}$ is *orientable.* (p. 76 [1])
- Any sufficiently small piece of a surface *isometric* to part of the *plane* is part of one of:
  - *Plane*
  - *Generalised cylinder*
  - *Generalised cone*
  - *Tangent developable*

  (p. 105 [1])
- A *diffeomorphism* $f : \mathcal{S}_1 \to \mathcal{S}_2$ is *conformal* $\Leftrightarrow$ for any surface patch $\overrightarrow{\sigma}_1$ of $\mathcal{S}_1$, the *first fundamental forms* of $\overrightarrow{\sigma}_1$ and $f \circ \overrightarrow{\sigma}_1$ are proportional (i.e. $E_1 = \lambda E_2, F_1 = \lambda F_2, G_1 = \lambda G_2$ for some smooth function λ). (thm. 5.2 [1])
- Let $\overrightarrow{\sigma}(u,v) = (f(u,v), g(u,v), 0)$ where f, g *smooth functions* in the uv-plane. Then: $\overrightarrow{\sigma}$ *conformal* $\Leftrightarrow$ ($f_u = g_v$ and $f_v = -g_u$) or ($f_u = -g_v$ and $f_v = g_u$). (exc. 5.14 [1])
- $\|\overrightarrow{\sigma}_u \times \overrightarrow{\sigma}_v\| = \sqrt{EG - F^2}$. (prop. 5.2 [1])
- $\mathcal{A}_{\overrightarrow{\sigma}}(R) = \int\int_R \sqrt{EG - F^2} dudv$ for a *regular surface.* Sometimes called $\int\int_R d\mathcal{A}_{\overrightarrow{\sigma}}$ (p. 113 [1])
- The *surface area* of a surface patch is *unchanged by reparametrisation.* (prop. 5.3 [1])
- A *diffeomorphism* is *equiareal* $\Leftrightarrow$ for any surface patch $\overrightarrow{\sigma}$ of $\mathcal{S}$, the first fundamental forms for patches $\overrightarrow{\sigma}$ on $\mathcal{S}_1$ and $f \circ \overrightarrow{\sigma}$ on $\mathcal{S}_2$ satisfy: $E_1 G_1 - F_1^2 = E_2 G_2 - F_2^2$. (thm. 5.3 [1])
- f *isometry* $\Rightarrow f$ *conformal map.* (exc. 5.9 [1])
- f *isometry* $\Rightarrow f$ *equiareal.* (exc. 5.18 [1])
- f *equiareal* and *conformal* $\Rightarrow$ *isometry.* (exc. 5.19 [1])
- A *connected* surface where all points are *umbilic*:
  Is either part of a *sphere* or a *plane.* (prop. 6.5 [1])

- Let P be a *non-umbilic* point of $\mathcal{S}$. Then there is a surface patch $\overrightarrow{\sigma}(u,v)$ of $\mathcal{S}$ containing P where $F = M = 0$ in the *1st* and *2nd fundamental forms*. I.e. the 1st and 2nd fundamental forms are: $Edu^2 + Gdv^2$ and $Ldu^2 + Ldv^2$ respectively for *smooth functions* E, G, L, N.
  Further more $\overrightarrow{\sigma}_u$ and $\overrightarrow{\sigma}_v$ are *principal vectors* with $\kappa_1 = L/E, \kappa_2 = N/G$. We call $\overrightarrow{\sigma}$ a *principal patch.* (prop. 7.2, p. 155-156 [1])
- Let P be a *hyperbolic* point of $\mathcal{S}$. Then there *exists* a *patch* of $\mathcal{S}$ containing P whose *parameter curves* are *asymptotic* curves. (exc. 7.14 [1])
- The *gaussian curvature* of a surface is *preserved by isometries.* This also means that the gaussian curvature is an *intrinsic* property of a surface. There is also a lemma 10.1 and some notation p. 230-231 used in the proof. (thm. 10.1, Gauss's Theorema Egregium (latin for "remarkable theorem") [1])
- For any integer $g \geq 0$, T_g can be given an *atlas* making it a *smooth surface.* Every *compact surface* is *diffeomorphic* to one of the T_g. (thm. 11.3 [1])

7.0.15 For Quadric surfaces $\mathcal{S}$:

- By applying a *rigid motion* of $\mathbb{R}^3$, every *non-empty quadric* in which the *coefficients* are *not all zero* can be transformed into one whose *cartesian equation* is *one of the following*:

  1. *Ellipsoid*: $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$
  2. *Hyperboloid of one sheet*: $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$
  3. *Hyperboloid of two sheets*: $\frac{z^2}{r^2} - \frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$
  4. *Elliptic paraboloid*: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = z$
  5. *Hyperbolic paraboloid*: $\frac{x^2}{p^2} - \frac{y^2}{q^2} = z$
  6. *Quadric cone*: $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 0$
  7. *Elliptic cylinder*: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$
  8. *Hyperbolic cylinder*: $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$
  9. *Parabolic cylinder*: $\frac{x^2}{p^2} = y$
  10. *Plane*: $x = 0$
  11. *Two parallel planes*: $x^2 = p^2$
  12. *Two intersecting planes*: $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 0$
  13. *Straight line*: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 0$
  14. *Single point*: $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 0$

  In all cases $p, q, r \neq 0$ are constants. (prop. 4.6 [1])
  Method:
  Diagonalize A and let the diagolalization matrix be P (a rotation matrix).

$\overrightarrow{r}' = (x', y', z') = (x, y, z)P$, $\overrightarrow{b}' = (b_1', b_2', b_3') = (b_1, b_2, b_3)P$.

Let $\begin{bmatrix} a_1' & 0 & 0 \\ 0 & a_2' & 0 \\ 0 & 0 & a_3' \end{bmatrix} = P^t A P$

Equation becomes: $a'(x')^2 + a_2'(y')^2 + a_3'(z')^2 + b_1'x' + b_2'y' + b_3'z' + c = 0$. For each $a_1', a_2', a_3' \neq 0$ we can eliminate b_1', b_2', b_3' respectively be translation etc.

- *Quadric* contains 3 *points on a line* $\Rightarrow$ it contains the *whole line.* (exc. 4.26 [1])
- L_1, L_2, L_3 *non-intersecting lines* in $\mathbb{R}^3$ $\Rightarrow$ there *exists a quadric* containing *all lines.* (exc. 4.26 [1])
- Any *doubly ruled surface* is part of a *quadric surface.* (exc. 4.27 [1])
- *Doubly ruled quadrics*: Hyperbolic paraboloid. (exc. 4.25, 4,27 [1])
- *Generalised cylinders*: Types 7-11. (exc. 4.24 [1])
- *Generalised cones*: Quadric cone. (exc. 4.24 [1])
- *Ruled surfaces*: Types 2, 5-10. (exc. 4.24 [1])
- *Surfaces of revolution*: Types 1-4, 6-7, 10. (exc. 4.24 [1])

7.0.16 For smooth tangent vector field on surfaces $\mathcal{S}$:

- If $\overrightarrow{V}$ is a *smooth tangent vector field* on a *compact surface* $\mathcal{S}$ with *finitely many stationary points* $P_1, P_2, \ldots, P_n$ then: $\sum_{i=1}^{n} \mu(P_i) = \chi$. (thm. 11.7 [1])
- A *tangent vector field* $\overrightarrow{V}$ is *smooth* $\Leftrightarrow$ the *three components* of $\overrightarrow{V}$ are *smooth functions* of (u, v). (exc. 11.14 [1])

7.0.17 For *smooth* surfaces $\mathcal{S}$:

- The *transition maps* of a *smooth surface* are *smooth.* (prop. 4.1 [1])
- If $\overrightarrow{\sigma} : U \to \mathcal{S} \cap W$ and $\overrightarrow{\widetilde{\sigma}} : \widetilde{U} \to \mathcal{S} \cap \widetilde{W}$ are 2 *allowable surface patches* of a *smooth surface* $\mathcal{S}$ and if $V \underset{open}{\subseteq} U, \widetilde{V} \underset{open}{\subseteq} \widetilde{U}$ such that $\overrightarrow{\sigma}(V) = \overrightarrow{\widetilde{V}} = \mathcal{S} \cap W \cap \widetilde{W}$, then $\Phi = \overrightarrow{\sigma}^{-1} \circ \overrightarrow{\widetilde{\sigma}} : \widetilde{V} \to V$ is *bijective*, *smooth* and has *smooth inverse.* Thus $\overrightarrow{\widetilde{\sigma}}$ is a *reparametrisation* of $\overrightarrow{\sigma}$ where they are both defined. (p. 69 [1])
- *Principle*: We can define *any property* of *any smooth surface* provided that we can define it for any *regular surface patch* in such a way that it is *unchanged* when the patch is *reparametrised.* (p. 69 [1])
- Let $\mathcal{S} \subseteq \mathbb{R}^3$ where *for all points* $P \in \mathcal{S}$ there *exists* $W \underset{open}{\subseteq} \mathbb{R}^3$ containing P and a *smooth function* $f : W \to \mathbb{R}$ such that:
  - $\mathcal{S} \cap W = \{(x, y, z) \in W \mid f(x, y, z) = 0\}$
  - The partial derivatives f_x, f_y, f_z do not all vanish at P

Then $\mathcal{S}$ is a *smooth surface.* (Thm. 4.1 [1])

- $f(x,y)$ *smooth function* $\Rightarrow$ *it's graph* $\{(x,y,x) \in \mathbb{R}^3 \mid z = f(x,y)\}$ is a *smooth surface* with atlas consisting of the patch $\overrightarrow{\sigma}(u,v) = (u,v,f(u,v))$. (this therem is repeated in the representation section of this document) (exc. 4.6 [1])
- *All components* of the *inclusion map* for $\mathcal{S}, f : \mathcal{S} \to \mathbb{R}^3$ are *smooth functions* $f_i : \mathcal{S} \to \mathbb{R}$. (exc. 4.11 [1])
- *Translations* and *invertible linear transformations* of $\mathbb{R}^3$ take *smooth surfaces* to *smooth surfaces.* (exc. 4.12 [1])

7.0.18 For *flat* surfaces $\mathcal{S}$:

- Let P be a *non-umbilic* point of $\mathcal{S}$, then there *exists* a *patch* of $\mathcal{S}$ that is a *ruled* surface. (prop. 7.3 [1])
- For all points P on $\mathcal{S}$ there *exists* a *neigborhood* around P where $\mathcal{S}$ is part of a *generalised cylinder*, a *generalised cone* or a *tangent developable.* (p. 157-158 [1])

7.0.19 For *parametrised* surfaces:

$\overrightarrow{\sigma} : U \to \mathbb{R}^3$.

- Let $\overrightarrow{\widetilde{\sigma}} : \widetilde{U} \to \mathbb{R}^3$ be a *surface patch* and suppose that for all $(\widetilde{u},\widetilde{v}) \in \widetilde{U}$ we are given *tangent vectors* $\overrightarrow{e}_1(\widetilde{u},\widetilde{v}) = a(\widetilde{u},\widetilde{v})\overrightarrow{\widetilde{\sigma}}_{\widetilde{u}} + b(\widetilde{u},\widetilde{v})\overrightarrow{\widetilde{\sigma}}_{\widetilde{v}}$, $\overrightarrow{e}_2(\widetilde{u},\widetilde{v}) = c(\widetilde{u},\widetilde{v})\overrightarrow{\widetilde{\sigma}}_{\widetilde{u}} + d(\widetilde{u},\widetilde{v})\overrightarrow{\widetilde{\sigma}}_{\widetilde{v}}$ whose components a,b,c,d are *smooth functions* of $(\widetilde{u},\widetilde{v})$. Assume that at some point $(\widetilde{u}_0,\widetilde{v}_0) \in \widetilde{U}$, the vectors $\overrightarrow{e}_1(\widetilde{u}_0,\widetilde{v}_0)$ and $\overrightarrow{e}_2(\widetilde{u}_0,\widetilde{v}_0)$ are *linearly independent.* Then there *exists* $\widetilde{V} \underset{open}{\subseteq} \widetilde{U}$ with $(\widetilde{u}_0,\widetilde{v}_0) \in \widetilde{V}$ and there *exists* a *reparametrisation* $\overrightarrow{\sigma}(u,v)$ of $\overrightarrow{\widetilde{\sigma}}(\widetilde{u},\widetilde{v})$, for $(\widetilde{u},\widetilde{v}) \in \widetilde{V}$ such that $\overrightarrow{\sigma}_u$ and $\overrightarrow{\sigma}_v$ are *parallel* to $\overrightarrow{e}_1$ and $\overrightarrow{e}_2$ respectively. (prop. 7.4 [1])
- Assume there is a $C \in \mathbb{R}$ such that $|\kappa_1| \leq C, |\kappa_2| \leq C$ everywhere on U. Let $\lambda \in \mathbb{R}$ be $|\lambda| < 1/C$ and let $\overrightarrow{\sigma}^{\lambda}$ be the corresponding parallel surface of $\overrightarrow{\sigma}$. Then:
  - $\overrightarrow{\sigma}^{\lambda}$ is a *regular* surface patch
  - The *standard unit normal* of $\overrightarrow{\sigma}^{\lambda}$ is the *same* as that for $\overrightarrow{\sigma}$
  - The *principal curvatures* of $\overrightarrow{\sigma}^{\lambda}$ are $\kappa_1/(1-\lambda\kappa_1)$ and $\kappa_2/(1-\lambda\kappa_2)$ and the corresponding *principal vectors* are the *same* as those of $\overrightarrow{\sigma}$ for κ_1 and κ_2 respectively
  - The *gaussian* and *mean* curvatures of $\overrightarrow{\sigma}^{\lambda}$ are: $\frac{K}{1-2\lambda H+\lambda^2 K}$ and $\frac{H-\lambda K}{1-2\lambda H+\lambda^2 K}$ respectively.

  (prop. 7.5 [1])

- Let $\overrightarrow{\tilde{\sigma}}(\tilde{u}, \tilde{v})$ be a *reparametrisation* of $\overrightarrow{\sigma}(u, v)$ with *reparametrisation map* $(u, v) = \Phi(\tilde{u}, \tilde{v})$, then:
  $\begin{bmatrix} \widetilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{bmatrix} = \pm \mathcal{J}^t \begin{bmatrix} L & M \\ M & N \end{bmatrix} \mathcal{J}$, where $\mathcal{J}$ is the *Jacobian* of Φ and we take the $+$ sign if $\det(\mathcal{J}) > 0$ and the $-$ sign if $\det(\mathcal{J}) < 0$. (exc. 6.3 [1])

- The *gaussian curvature* is given by:
  $$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG-F^2)^2}$$
  (cor. 10.1 [1])

- If $F = 0$ we have $K = -\frac{1}{2\sqrt{EG}}(\frac{\partial}{\partial u}(\frac{G_u}{\sqrt{EG}}) + \frac{\partial}{\partial v}(\frac{E_v}{\sqrt{EG}}))$. (cor. 10.2 [1])

- If $E = 1$ and $F = 0$ we have $K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2}$. (cor. 10.2 [1])

- Any point of a surface of *constant gaussian curvature* is contained in a *patch* that is *isometric* to part of a *plane*, a *sphere* or a *pseudosphere.* (thm. 10.2 [1])

- If $\overrightarrow{\sigma}$ has *constant mean curvature* $H \neq 0$, then for $\lambda = 1/2H$, $\overrightarrow{\sigma}^{\lambda}$ has *constant gaussian curvature* $4H^2$.
  Conversely, *constant positive gaussian curvature* $K \Rightarrow$ for $\lambda = \pm 1/\sqrt{K}$, $\overrightarrow{\sigma}^{\lambda}$ has *constant mean curvature* $\mp\frac{1}{2}\sqrt{K}$. (cor. 7.1 [1])

- If $\mathcal{S}$ is a *compact* surface (i.e. $\mathcal{S} \underset{compact}{\subseteq} \mathbb{R}^3$), then there *exists* a point P of $\mathcal{S}$ where the *gaussian curvature* $K > 0$. (prop. 7.6 [1])

- Every *compact* surface whose *gaussian curvature* is *constant* is a *sphere.* (thm. 10.4 [1])

- A *compact surface* with $K > 0$ everywhere and *constant mean curvature* is a *sphere.* (exc. 10.12 [1])

- Let $P = \overrightarrow{\sigma}(u_0, v_0)$ be a *non-umbilic point.* Let $\kappa_1 \geq \kappa_2$ be the *principal curvatures* of $\overrightarrow{\sigma}$ and suppose that κ_1 has a *local maximum* at P and κ_2 has a *local minimum* at P. Then $K \leq 0$ at P. (lemma 10.2 [1])

- Assume that *1st fundamental form* of $\overrightarrow{\sigma}$ is of the form $E(du^2 + dv^2)$ (i.e. $E = G, F = 0$). Then $\overrightarrow{\sigma}_{uu} + \overrightarrow{\sigma}_{vv}$ is *perpendicular* to $\overrightarrow{\sigma}_u$ and $\overrightarrow{\sigma}_v$.
  The *mean curvature* $H = 0$ everywhere $\Leftrightarrow$ the *laplacian* $\overrightarrow{\sigma}_{uu} + \overrightarrow{\sigma}_{vv} = 0$. (exc. 7.15 [1])

- Let the *1st* and *2nd fundamental forms* of $\overrightarrow{\sigma}$ have $F = 0, M = 0$. Define $\overrightarrow{\Sigma}(u, v, w) = \overrightarrow{\sigma}(u, v) + w\overrightarrow{N}(u, v)$. The 3 families of surfaces obtained by *fixing* u, v or w respectively in $\overrightarrow{\Sigma}$ form a *triply orthogonal system.*
  The surfaces with w constant are parallel to $\overrightarrow{\sigma}$. (exc. 7.17 [1])

- The *area* of the *image* $\overrightarrow{\sigma}(R)$ *under the Gauss map* where $R \subseteq U$ is: $\int\int_R |K| d\mathbb{A}_{\overrightarrow{\sigma}}$. (exc. 7.18 [1])

- Let $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ and $\overrightarrow{\tilde{\sigma}} : U \to \mathbb{R}^3$ be surface *pathces* with the *same first* and *second fundamental forms.* Then, there is a *rigid motion* M of $\mathbb{R}^3$ such that $\overrightarrow{\tilde{\sigma}} = M \circ \overrightarrow{\sigma}$.
Moreover let $V \underset{open}{\subseteq} \mathbb{R}^3$ and let E, F, G, L, M, N be *smooth functions* on V. Assume that $E > 0, G > 0, EG - F^2 > 0$ and that the equations in Corollary 10.1 and Proposition 10.4 hold, with $K = \frac{LN-M^2}{EG-F^2}$ and the Cristoffel symbols defined. Then if $(u_0, v_0) \in V$ there *exists* an $U \underset{open}{\subseteq} V$ where $(u_0, v_0) \in U$ (i.e. U is an open neigborhood in V around (u_0, v_0)) and a surface patch $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ such that $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$ are the first and second fundamental forms of $\overrightarrow{\sigma}$ respectively. (thm. 10.3 [1])

7.0.20 Critical Points

Let $\mathcal{S}$ be a *smooth* surface:

- If F is a *smooth function* on $\mathcal{S}$, then there *exists* a *unique smooth tangent vector field* $\nabla_{\mathcal{S}} F$ on $\mathcal{S}$ such that if P is a point of $\mathcal{S}$ and $\overrightarrow{\gamma}(t)$ is a curve in $\mathcal{S}$ which passes through P when $t = 0$, we have $(\nabla_{\mathcal{S}} F).\overrightarrow{\gamma}'(0) = \frac{d}{dt}|_{t=0} F(\overrightarrow{\gamma}(t))$. Moreover, P is a critical point of F if and only if $\nabla_{\mathcal{S}} F = \overrightarrow{0}$ at P. (prop. 11.1 [1])

- Let P be a *critical point* of a *smooth function* F on $\mathcal{S}$. Then the multiplicity of P as a stationary point of $\nabla_{\mathcal{S}} F$ is:
$$\mu(P) = \begin{cases} 1 & if\ P\ is\ a\ local\ maximum\ or\ a\ local\ minimum \\ -1 & if\ P\ is\ a\ saddle\ point \end{cases}$$
(prop. 11.2 [1])

- Let $F : \mathcal{S} \to \mathbb{R}$ be a *smooth function* on a *compact surface* $\mathcal{S}$ with only *finitely many critical points* all *non-degenerate.* Then:
$(number\ of\ local\ maxima\ of\ F) - (number\ of\ saddle\ points\ of\ F) + (number\ of\ local\ minima\ of\ F) = \chi$,
the Euler number of $\mathcal{S}$. (thm. 11.8 [1])

8 Global Pointwise Local Surface Results

- The *unit sphere* $\mathcal{S}^2$ given by $f(x, y, z) = x^2 + y^2 + z^2 - 1$ is *smooth*, where $f : \mathbb{R}^3 \to \mathbb{R}$. (ex. 4.7 [1])

- The *unit sphere cannot* be parametrised by a *single patch.* (exc. 4.5 [1])

- The *double cone* given by $f(x, y, z) = x^2 + y^2 - z^2$ is *smooth*, where $f : (\mathbb{R}^3 \backslash \{(0, 0, 0)\}) \to \mathbb{R}$. (ex. 4.7 [1])

- For the *surface of revolution* $\overrightarrow{\sigma}(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$:
All points are *parabolic* $\Leftrightarrow$ $\overrightarrow{\sigma}$ is part of a *circular cylinder* or a *circular cone.* (exc. 6.23 [1])
Assuming $f > 0$ and $(f')^2 + (g')^2 = 1$ everywhere: $K = -\frac{f''}{f}$. (ex. 7.2, 10.1 [1])

- p, q, r *distinct positive* numbers $\Rightarrow$ there are exactly 4 *umbilics* on the *ellipsoid* $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$. (exc. 6.24 [1])
- Let $\mathcal{S}_+$ and $\mathcal{S}_+$ be the parts of the *torus* where K is *positive* and *negative* respectively. Then $\int\int_{\mathcal{S}_+} K d\mathcal{A} = -\int\int_{\mathcal{S}_-} K d\mathcal{A} = 4\pi$. (exc. 7.19 [1])
- If $\mathcal{S}$ has first fundamental form $e^{\lambda}(du^2 + dv^2)$, where λ is a smooth function of u and v, then the gaussian curvature satisfies $\Delta\lambda + 2Ke^{\lambda} = 0$, where Δ denotes the laplacian $\partial^2/\partial u^2 + \partial^2/\partial v^2$. (exc. 10.1 [1])
- *Any* map of *any* region of the *earth*'s surface *must distort distances.* (prop. 10.1 [1])
- The *only isometries* of a *helicoid* $\vec{\sigma}(u, v) = (u \cos v, u \sin v, v)$ are $S_\lambda, R_x \circ S_\lambda, R_y \circ S_\lambda$ and $R_z \circ S_\lambda$ for some value λ, where S_λ is the *screwing motion* $\vec{\sigma}(u, v) \mapsto \vec{\sigma}(u, v + \lambda)$ and R_x, R_y, R_z are *rotations* by π around x, y, z-axes. (prop. 10.2 [1])
- There are *no isometries* between *any region* of a *sphere* and *any region* of a *generalised cylinder* or a *generalised cone.* (exc. 10.5 [1])
- The *gaussian curvature* of the *Móbius band* is $-\frac{1}{4}$ everywhere along it's *meridian cicle*, so it *cannot* be constructed by taking a strip of paper and joining the ends together with a half-twist. (exc. 10.6 [1])
- The *gaussian curvature* of the following two patches $\vec{\sigma}$ and $\vec{\vec{\sigma}}$ are the *same*, but there is *no isometry* from $\vec{\sigma}$ to $\vec{\vec{\sigma}}$: $\vec{\sigma} = (u \cos v, u \sin v, \ln u)$ and $\vec{\vec{\sigma}} = (u \cos v, u \sin v, v)$. (exc. 10.7 [1])
- The *only isometries* from the *catenoid* to *itself* are *products* of *rotations* around it's axis, *reflections* in planes containing the axis, and *reflection* in the plane containing the waist of the catenoid. (exc. 10.8 [1])
- If the *first* and *second fundamental forms* of a surface are $du^2 + dv2$ and $-du^2$ respectively, then the *surface is given* by the patch $\vec{\sigma}(u, v) = (\cos u, \sin u, \lambda v)$, which is the parametrisation of a *circular cylinder* of *radius* 1. (ex. 10.2 [1])
- $E = \cos^2 v, F = 0, G = 1, L = -\cos^2 v, M = 0, N = -1 \Rightarrow$ surface is part of a *sphere* of *radius* 1. (exc. 10.9 [1])
- *Second fundamental form* of $\vec{\sigma}$ is *zero everywhere* $\Rightarrow$ $\vec{\sigma}$ is part of a *plane.* (exc. 6.2 [1])
- A *compact* surface whose *gaussian curvature* K is > 0 everywhere is *diffeomorphic* to a *sphere.* (exc. 11.11 [1])
- *Triply orthogonal systems* of familes a, b, c in each case:
  - a, b, c: Planes parallel to one of the 3 coord. planes. (p. 90 [1])
  - a: Spheres with center $(0, 0, 0)$, b: Planes containing the z-axis, c: Circular cones with z-axis as axis. (exc. 4.28 [1])
  - a: Planes parallel to the xy-plane, b: Planes containing the z-axis, c: Circular cones with z-axis as axis. (exc. 4.28 [1])

  - a: Ellipsoids, b: Hyperboloids of one sheet, c: Hyperboloids of 2 sheets. (p. 90 [1])
  - a, b: Elliptic paraboloids, c: hyperbolic paraboloids. (exc. 4.29 [1])

- Given the *catenoid* with one meridian removed: $\overrightarrow{\sigma}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$ for $0 < v < 2\pi$ and the part of the *helicoid* between the planes $z = 0$ and $z = 2\pi$: $\overrightarrow{\sigma}(u,v) = (u \cos v, u \sin v, v)$ for $0 < v < 2\pi$, then the *map* from the catenoid to the helicoid that takes $\overrightarrow{\sigma}(u,v)$ to $\overrightarrow{\tilde{\sigma}}(\sinh u, v)$ is an *isometry.* (exc. 5.8 [1])

- There is a *isometric deformation* of the *catenoid* to the *helicoid.* (exc. 5.8 [1])

8.0.21 Smooth tangent vector field results

- *Source*: $\overrightarrow{V}(x,y) = (x,y)$, $\mu = +1$. (ex. 11.1 [1])
- *Sink*: $\overrightarrow{V}(x,y) = (-x,-y)$, $\mu = +1$. (ex. 11.1 [1])
- *Vortex*: $\overrightarrow{V}(x,y) = (y,-x)$, $\mu = +1$. (ex. 11.1 [1])
- *Bifurcation*: $\overrightarrow{V}(x,y) = (x,-y)$, $\mu = -1$. (ex. 11.1 [1])
- For $k \in \mathbb{Z} \setminus \{0\}$ let $\overrightarrow{V} = (\alpha,\beta)$ be the vector field in the plane given by $\alpha + i\beta = \begin{cases} (x+iy)^k & if k > 0 \\ (x-iy)^{-k} & if k < 0 \end{cases}$.
Then $(0,0)$ is a *stationary point* with multiplicity $\mu = k$. (exc. 11.13 [1])

9 Curve on Surface Properties

9.0.22 Notation for these curve on surface sections:

- $\vec{\sigma}$: Surface patch
- $\vec{\gamma}$: Curve on surface
- $\vec{n}$ is principal normal of $\vec{\gamma}$
- $\vec{N}$ is unit normal of $\vec{\sigma}$
- $\vec{t}$ is tangent vector of $\vec{\gamma}$
- $\vec{b}$ is binormal of $\vec{\gamma}$
- E, F, G is from the *first fundamental form* of $\vec{\sigma}$
- L, M, N is from the *second fundamental form* of $\vec{\sigma}$

9.0.23 For curves on surfaces $\mathcal{S}$:

$\vec{\gamma}(t)$.

- If $\vec{\gamma}(t) = \vec{\sigma}(u(t), v(t))$ is a *curve in a surface patch* $\vec{\sigma}$, it's *arc-length* starting at a point $\vec{\gamma}(t_0)$ is given by: $s = \int_{t_0}^{t} \|\vec{\gamma}'(u)\| du$. (p. 97 [1])
  $s = \int_{t_0}^{t} (E(u')^2 + 2Fu'v' + G(v')^2)^{1/2} dt = \int \sqrt{ds^2}$ where
  $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ (p. 98, First Fundamental Form [1])
  First fundamental form will change when *patch is changed.* (exc. 5.4 [1])
- $\|\vec{\gamma}'\|^2 = E(u')^2 + 2Fu'v' + G(v')^2$ where
  $E = \|\vec{\sigma}_u\|^2 = \vec{\sigma}_u . \vec{\sigma}_v$,
  $F = \vec{\sigma}_u . \vec{\sigma}_v$,
  $G = \|\vec{\sigma}_v\|^2 = \vec{\sigma}_v . \vec{\sigma}_v$. (p. 97-98 [1])
- If the curves $\vec{\gamma}$ and $\vec{\vec{\gamma}}$ on $\mathcal{S}$ *intersect* at point P that lies in a surface patch $\vec{\sigma}$ of $\mathcal{S}$. Then $\vec{\gamma}(t) = \vec{\sigma}(u(t), v(t))$ and $\vec{\vec{\gamma}}(t) = \vec{\sigma}(\tilde{u}(t), \tilde{v}(t))$ for some smooth functions $u, v, \tilde{u}, \tilde{v}$.
  For some parameter values $t_0, \tilde{t_0}$ we have $\vec{\sigma}(u(t_0), v(t_0)) = P = \vec{\sigma}(\tilde{u}(\tilde{t_0}), \tilde{v}(\tilde{t_0}))$.
  The *angle θ of intersection* of $\vec{\gamma}$ and $\vec{\vec{\gamma}}$ at $P \underset{def}{=}$
  the angle between the *tangent vectors* $\vec{\gamma}'$ and $\vec{\vec{\gamma}}'$ (evaluated at $t = t_0$ and $t = \tilde{t_0}$ respectively).
  $\cos\theta = \frac{\vec{\gamma}' . \vec{\vec{\gamma}}'}{\|\vec{\gamma}'\| \|\vec{\vec{\gamma}}'\|} = \frac{Eu'\tilde{u}' + F(u'\tilde{v}' + \tilde{u}'v') + Gv'\tilde{v}'}{\sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\sqrt{E(\tilde{u}')^2 + 2F\tilde{u}'\tilde{v}' + G(\tilde{v}')^2}}$. (p. 107 [1])
- *Parameter curves* on a surface patch $\vec{\sigma}(u, v) \underset{def}{=}$
  $\vec{\gamma}(t) = \vec{\sigma}(a, t)$, $\vec{\vec{\gamma}}(t) = \vec{\sigma}(t, b)$
  where a is a *constant* u value and b is a *constant* v value. (ex. 5.6 [1])
- *Angle of intersection* of the *parameter curves* at $\vec{\sigma}(a, b) \underset{def}{=}$
  $\cos\theta = \frac{F}{\sqrt{EG}}$. They are *orthogonal* $\Leftrightarrow F = 0$. (ex. 5.6 [1])

- $\vec{\gamma}$ is a *geodesic* $\underset{def}{=}$ $\vec{\gamma}''(t)$ is *zero* or *perpendicular* to $\mathcal{S}$ at the point $\vec{\gamma}(t)$ - i.e. parallel to $\vec{N}$ for all t. (def. 8.1 [1])
- $\vec{\gamma}(t) = \vec{\sigma}(u(t), v(t))$ on surface patch $\vec{\sigma} : R \to \mathbb{R}^3$ is called a *simple closed curve* with *period a* $\underset{def}{=}$ $\vec{\pi}(t) = (u(t), v(t))$ is a *simple closed curve* in $\mathbb{R}^2$ with period a such that the region $int(\vec{\pi})$ of $\mathbb{R}^2$ enclosed by $\vec{\pi}$ is *entirely contained* in U.
  Furthermore $\vec{\gamma}$ is *positively oriented* $\underset{def}{=}$ $\vec{\pi}$ is *positively oriented.*
  Finally the *interior* of $\vec{\gamma}$ $int(\vec{\gamma})$ $\underset{def}{=}$ the *image* of $int(\vec{\pi})$ *under the map* $\vec{\sigma}$. (def. 11.1 [1])

9.0.24 For *unit-speed* curves on surfaces:

$\vec{\gamma}(t) = \vec{\sigma}(u(t), v(t))$.

- The *normal curvature* κ_n and *geodesic curvature* κ_g of $\vec{\gamma}$ $\underset{def}{=}$ $\vec{\gamma}'' = \kappa_n \vec{N} + \kappa_g \vec{N} \times \vec{\gamma}'$. Hence:
  - *normal curvature* $\kappa_n = \vec{\gamma}''.\vec{N}$
  - *geodesic curvature* $\kappa_g = \vec{\gamma}''.(\vec{N} \times \vec{\gamma}')$

  (p. 127 [1])
- The *curvature* κ of $\vec{\gamma}$ $\underset{def}{=} \sqrt{\kappa_n^2 + \kappa_g^2}$. (p. 127 [1])
- $\vec{\gamma}$ is a *normal section* of the surface $\underset{def}{=}$ $\vec{\gamma}$ is the *intersection* of the surface with a *plane* Π that is *perpendicular* to the *tangent plane* at every point of $\vec{\gamma}$. (p. 128 [1])
- The *geodesic torsion* τ_g of $\vec{\gamma}$ $\underset{def}{=} \tau_g = \tau + \psi'$ where $\kappa > 0$ and ψ is the angle between $\vec{\gamma}''$ and $\vec{N}$. (exc. 6.11 [1])
- $\vec{\gamma}$ is *asymptotic* on $\mathcal{S}$ $\underset{def}{=}$ *normal curvature* of $\vec{\gamma}$ is *everywhere zero.* (exc. 6.12 [1])
- Let $\vec{t}_1 = \xi_1 \vec{\sigma}_u + \eta_1 \vec{\sigma}_v$, $\vec{t}_2 = \xi_2 \vec{\sigma}_u + \eta_2 \vec{\sigma}_v$ and $T_1 = \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix}$, $T_2 = \begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix}$ then:
  $\vec{t}_1.\vec{t}_2 = \begin{bmatrix} \xi_1 & \eta_1 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix} = T_1^t \mathcal{F}_I T_2$. (p. 132 [1])
- If $T = \begin{bmatrix} u' \\ v' \end{bmatrix}$ where $\vec{\gamma}' = u' \vec{\sigma}_u + v' \vec{\sigma}_v$ then:
  $\kappa_n = T^t \mathcal{F}_{II} T$. (p. 132 [1])
- A curve $\mathcal{C}$ on $\mathcal{S}$ is a *line of curvature* $\underset{def}{=}$ the *tangent vector* of $\mathcal{S}$ is a *principal* vector of $\mathcal{S}$ at all points of $\mathcal{C}$. (exc. 6.18 [1])
- *Geodesic coordinates*: The parametrisation of a patch $\vec{\sigma}(u, v) = \vec{\gamma}^{v}(u)$ where $\vec{\gamma}^{v}(u)$ is the *unique unit-speed geodesic* such that $\vec{\gamma}^{v}(0) = \vec{\gamma}(v)$ and which is *perpendicular* to $\vec{\gamma}$ at $\vec{\gamma}(v)$, where $\vec{\gamma}(v)$ is some *given unit-speed geodesic.* (p. 197 [1])

9.0.25 For *curviliear polygons* on surfaces:

- $\vec{\gamma} = \vec{\sigma} \circ \vec{\pi}$ is a *curvilinear polygon* on the *surface patch* $\vec{\sigma} \underset{def}{=} \vec{\pi} : \mathbb{R} \to U$ is a *curvilinear polygon* in U. (p. 253 [1])
- $int(\vec{\gamma}$ $\underset{def}{=}$ the *image under* $\vec{\sigma}$ of $int(\vec{\pi}$.
- Let $\theta_i^{\pm}$ be the *angles* between $(\vec{\gamma}')^{\pm}(t_i)$ and *some vector* $\vec{e_a}$ *perpendicular* to $\vec{N}$, then:
  - $\delta_i = \theta_i^{+} - \theta_i^{-}$ is the *external angle* at the *vertex* $\vec{\gamma}(t_i)$
  - $\alpha_i = \pi - \delta_i$ is the *internal angle* at the *vertex* $\vec{\gamma}(t_i)$ (and we assume $0 < \alpha_i < 2\pi$ for $i = 1, \ldots, n$)

  (p. 253 [1])
- A curvilinear polygon $\vec{\gamma}$ is *unit-speed* $\underset{def}{=}$ $\|\vec{\gamma}'\| = 1$ for all t where $\vec{\gamma}(t)$ is *not a vertex.* (p. 253 [1])
- The *length* of the curvilinear polygon $\vec{\gamma}$ $\underset{def}{=}$ the *sum* of the *lengths* of the *edges* of $\vec{\gamma}$. (p. 253 [1])
- If $\mathcal{S}$ is a surface with atlas $\vec{\sigma}_i : U_i \to \mathbb{R}^3$, then a *triangulation of* $\mathcal{S}$ $\underset{def}{=}$ a *collection of curvilinear polygons*, the interior of each of which is contained in one of the $\vec{\sigma}_i(U_i)$ such that:
  - *Every point* of $\mathcal{S}$ is *in at least one* of the *curvilinear polygons*
  - Two *curvilinear polygons* are *either disjoint or* their *intersection* is a *common edge* or a *common vertex*
  - Each *edge* is an edge of *exactly 2 polygons*

  (def. 11.3 [1])
- The *Euler number* χ of a *triangulation* of a *compact surface* $\mathcal{S}$ $\underset{def}{=}$ $\chi = V - E + F$, where V is *total number of vertices*, E is *total number of edges*, F is *total number of polygons* in the triangulation. (def. 11.4 [1])
- The *area* of a *triangulation* of $\mathcal{S}$ with polygons P_i $\underset{def}{=}$ $\int\int_{\mathcal{S}} K d\mathcal{A} = \sum_i \int\int_{R_i} K d\mathcal{A}_{\vec{\sigma}_i}$ where each P_i is contained in the image of some patch $\vec{\sigma}_i : U_i \to \mathbb{R}^3$ in the atlas of $\mathcal{S}$, say $P_i = \vec{\sigma}_i(P_i)$, where $R_i \subseteq U_i$. (p. 261 [1])

10 Curve on Surface Theorems

10.0.26 For *unit-speed* curves on surfaces:

$\vec{\gamma}(t) = \vec{\sigma}(u(t), v(t))$.

- If $\vec{\gamma} :]\alpha, \beta[\to \mathbb{R}^3$ is a *curve* whose image is *contained in a surface patch* $\vec{\sigma} : U \to \mathbb{R}^3$, then $\vec{\gamma}(t) = \vec{\sigma}(u(t), v(t))$ for some *smooth map* $t \mapsto (u(t), v(t)) :]\alpha, \beta[\to U$. (exc. 4.30 [1])
- $\vec{\gamma}', \vec{N}$ and $\vec{N} \times \vec{\gamma}'$ are *mutually perpendicular unit vectors.* (p. 127 [1])
- $\kappa_n = \kappa \vec{n}.\vec{N} = \kappa \cos \psi$, where ψ is angle between $\vec{n}$ and $\vec{N}$. (p. 128 [1])
- $\kappa_g = \pm \kappa \sin \psi$, where ψ is angle between $\vec{n}$ and $\vec{N}$. (p. 128 [1])
- $\kappa_n = L(u')^2 + 2Mu'v' + N(v')^2$. (prop. 6.1 [1])
- κ_n and κ_g either *stay the same* or *both change sign* when $\vec{\sigma}$ is *reparametrised.* (p. 128 [1])
- When $\vec{\gamma}$ *changes parameter* to $\pm t + c$, where c constant: κ_n *unchanged* and κ_g *may change sign.* (p. 128 [1])
- *Two unit-speed curves* passing through P on a surface with the *same tangent vector* at P have the same *same normal curvature* at P. (p. 130 [1])
- $\vec{\gamma}$ *normal section* $\Rightarrow \kappa_n = \pm \kappa$ and $\kappa_g = 0$. (p. 128 [1])
- $\vec{\gamma}$ has $\kappa_n = \kappa_g = 0$ everywhere $\Rightarrow$ $\vec{\gamma}$ is part of a *straight line.* (exc. 6.6 [1])
- $\kappa_n = \pm 1/r$ for any curve on a *sphere* of radius r. (exc. 6.7 [1])
- If $\kappa > 0$ and $\vec{\gamma}$ form the *intersection between two surfaces* $\mathcal{S}_1$ and $\mathcal{S}_2$ with unit normals $\vec{N}_1, \vec{N}_2$. κ_1 and κ_2 are normal curvatures of $\vec{\gamma}$ when viewed as a curves in $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. Then:
  $\kappa_1 \vec{N}_2 - \kappa_2 \vec{N}_1 = \kappa(\vec{N}_1 \times \vec{N}_2) \times \vec{n}$ and
  $\kappa^2 \sin^2 \alpha = \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2 \cos \alpha$, where α is the angle between $\mathcal{S}_1$ and $\mathcal{S}_2$. (exc. 6.10 [1])
- Let $\kappa > 0$, ψ be the angle between $\vec{\gamma}''$ and $\vec{N}$, and let $\vec{B} = \vec{t} \times \vec{N}$ then:
  - $\vec{N} = \vec{n} \cos \psi + \vec{b} \sin \psi$
  - $\vec{B} = \vec{b} \cos \psi - \vec{n} \sin \psi$
  - $\vec{t}' = \kappa_n \vec{N} - \kappa_g \vec{B}$
  - $\vec{N}' = -\kappa_n \vec{t} + \tau_g \vec{B}$
  - $\vec{B}' = \kappa_g \vec{t} - \tau_g \vec{N}$
- Any *straight line* on $\mathcal{S}$ is an *asymptotic* curve. (exc. 6.12 [1])
- Let $\kappa > 0$. $\vec{\gamma}$ is *asymptotic* $\Leftrightarrow$
  $\vec{b}$ is *parallel* to *unit normal* of $\mathcal{S}$ for all points of $\vec{\gamma}$. (exc. 6.12 [1])

- An *asymptotic* curve with *positive curvature* has *torsion* equal to *geodesic torsion.* (exc. 6.14 [1])

- Let P be a point on $\mathcal{S}$ and $\overrightarrow{v}$ be a *unit-tangent* vector to $\mathcal{S}$ at P. Let Π_θ be the *plane through P parallel* to $\overrightarrow{v}$ and making *angle θ* with the *tangent plane* to $\mathcal{S}$. If Π_θ *intersects* $\mathcal{S}$ in a curve with *curvature κ_θ* then $\kappa_\theta \sin\theta$ is *independent* of θ. (prop. 6.2 [1])

- Let $\overrightarrow{\gamma}^\tau(t)$ be a *smooth family of curves* on $\overrightarrow{\sigma}$ passing through $\overrightarrow{p}$ and $\overrightarrow{q}$ for $\tau \in]-\delta, \delta[$ where:

  - There *exists* an $\epsilon > 0$ such that $\overrightarrow{\gamma}^\tau(t)$ is *defined* for all $t \in]-\epsilon, \epsilon[$ and all $\tau \in]-\delta, \delta[$
  - For some a, b with $-\epsilon < a < b < \epsilon$ we have $\overrightarrow{\gamma}^\tau(a) = \overrightarrow{p}$ and $\overrightarrow{\gamma}^\tau(a) = \overrightarrow{p}$ for all $\tau \in]-\delta, \delta[$
  - The *map from* the *rectangle* $]-\delta, \delta[\times]-\epsilon, \epsilon[$ *into* $\mathbb{R}^3$ given by given by $(\tau, t) \mapsto \overrightarrow{\gamma}^\tau(t)$ is *smooth*
  - $\overrightarrow{\gamma}^0 = \gamma$

  The *length* of the part of $\overrightarrow{\gamma}^\tau$ between $\overrightarrow{p}$ and $\overrightarrow{q}$ is $\mathcal{L}(\tau) = \int_a^b \|(\overrightarrow{\gamma}^\tau)'\| dt$. Then: The *unit-speed* curve $\overrightarrow{\gamma}$ is a *geodesic* if and only if $\frac{d}{d\tau}\mathcal{L}(\tau) = 0$ when $\tau = 0$ for *all families of curves* $\overrightarrow{\gamma}^\tau$ with $\overrightarrow{\gamma}^0 = \overrightarrow{\gamma}$. Note that although $\overrightarrow{\gamma} = \overrightarrow{\gamma}^0$ is assumed to be unit-speed, we do *not* assume that $\overrightarrow{\gamma}^\tau$ is unit-speed if $\tau \neq 0$. (thm. 8.2 [1])

- There is an $U \underset{open}{\subseteq} \mathbb{R}^2$ containing $(0,0)$ such that the parametrisation of *geodesic coordinates* $\overrightarrow{\sigma}(u,v) = \overrightarrow{\overrightarrow{\gamma}}^v(u) : U \to \mathbb{R}^3$ is an *allowable surface patch* for $\mathcal{S}$. Moreover, the first fundamental form of $\overrightarrow{\sigma}$ is $du^2 + G(u,v)dv^2$ where G is a *smooth function* on U such that $G(0,v) = 1$, $G_u(0,v) = 0$ whenever $(0,v) \in U$. (prop. 8.7 [1])

- Let P be a point of a surface $\mathcal{S}$ and let $\overrightarrow{v}$ be a unit tangent vector to $\mathcal{S}$ at P. Let $\overrightarrow{\gamma}^\gamma(r)$ be the unit-speed geodesic on $\mathcal{S}$ passing through P when $r = 0$ and with tangent vector $\overrightarrow{v}$ there.
  Define the *geodesic polar patch* $\overrightarrow{\sigma}(r, \theta) = \overrightarrow{\gamma}^\theta(r)$. $\overrightarrow{\sigma}$ is a smooth for all $-\epsilon < r < \epsilon$ and all values of θ where $\epsilon > 0$, and it is an allowable surface patch for $\mathcal{S}$ defined for $0 < r < \epsilon$ and for θ in any open interval of length $\leq 2\pi$. If $0 < R < \epsilon$ then $\int_0^R \|\frac{d\overrightarrow{\gamma}^\theta}{dr}\|^2 = R$. Furthermore $\overrightarrow{\sigma}_u . \overrightarrow{\sigma}_\theta = 0$ and the first fundamental form of $\overrightarrow{\sigma}$ is $dr^2 + G(r,\theta)d\theta^2$. (exc. 8.21, Gauss's Lemma, [1])

10.0.27 For *unit-speed simple closed* curves on surfaces:

$\overrightarrow{\gamma}(t)$.

- If $\overrightarrow{\gamma}$ is *positively oriented* with *length* $l(\overrightarrow{\gamma})$ then:
  $\int_0^{l(\overrightarrow{\gamma})} \kappa_g ds = 2\pi - \int\int_{int(\overrightarrow{\gamma})} K d\mathcal{A}_{\overrightarrow{\sigma}}$ where κ_g is geodesic curvature of $\overrightarrow{\gamma}$, K is the gaussian curvature of $\overrightarrow{\sigma}$ and $d\mathcal{A}_{\overrightarrow{\sigma}} = (EG - F^2)^{1/2} dudv$ is the area element on $\overrightarrow{\sigma}$. (thm. 11.1, Gauss-Bonnet for simple closed curves [1])

- Let $\{\overrightarrow{e}_a, \overrightarrow{e}_b, \overrightarrow{N}\}$ be a *right-handed orthonormal basis* of $\mathbb{R}^3$ where $\overrightarrow{N}$ is the *unit normal* to $\overrightarrow{\sigma}$. Let $\theta(s)$ be the *angle* between the *unit tangent vector* $\overrightarrow{\gamma}'$ of $\overrightarrow{\gamma}$ at $\overrightarrow{\gamma}(s)$ and the *unit vector* $\overrightarrow{e}_a$ at the *same point.* Then $\int_0^{l(\overrightarrow{\gamma})} \theta' ds = 2\pi$. (p. 250, Hopf's Umlaufsatz (German for Hopf's rotation theorem) [1])
- If $\overrightarrow{\gamma}(s)$ is *parametrised* by *arc-length* and has *total length* $l(\overrightarrow{\gamma})$ then $\int_0^{l(\overrightarrow{\gamma})} \kappa_s(s) ds = 2\pi$. (exc. 11.2 [1])
- If $\overrightarrow{\sigma}$ has *constant* $K \leq 0$ everywhere then there are *no simple closed geodesics* in $\overrightarrow{\sigma}$. (exc. 11.1 [1])

10.0.28 For *curviliear polygons* on surfaces:

- For *positively oriented unit-speed* curvilinear polygon $\overrightarrow{\gamma}$ with n *edges*: $\int_0^{l(\overrightarrow{\gamma})} \kappa_g ds = \sum_{i=1}^n \alpha_i - (n-2)\pi - \int\int_{int(\overrightarrow{\gamma})} K d\mathcal{A}_{\overrightarrow{\sigma}}$. (thm. 11.2, Gauss-Bonnet for curvilinear polygons [1]) Special cases of this theorem:
    - If the n *edges* are *geodesics*: $\sum_{i=1}^n \alpha_i = (n-2)\pi + \int\int_{int(\overrightarrow{\gamma})} K d\mathcal{A}_{\overrightarrow{\sigma}}$. (cor. 11.1 [1])
    - If $\overrightarrow{\sigma}$ is the *plane* and the *edges are lines*: $\sum_{i=1}^n \alpha_i = (n-2)\pi$. (p. 256 [1])
    - For a *triangle* of 3 *geodesic edges* on the *unit sphere*: $\mathcal{A}(ABC) = \angle A + \angle B + \angle C - \pi$. (p. 256, exc. 10.4, thm. 5.5 [1])
    - On the *pseudosphere* with 3 *geodesic edges*: $\mathcal{A}(ABC) = \pi - \angle A - \angle B - \angle C$. (p. 257 [1])
- If K of $\overrightarrow{\sigma}$ has $K \leq -1$ everywhere then a *curvilinear n-gon* on $\overrightarrow{\sigma}$ with *geodesic edges* have $n \geq 3$ and if $n = 3$, then the area enclosed in $\overrightarrow{\gamma}$ cannot exceed π. (exc. 11.3 [1])
- Every *compact surface* has a *triangulation* with *finitely* many polygons. (thm. 11.4 [1])
- For *any triangulation* of a *compact surface* $\mathcal{S}$:
  $\int\int_{\mathcal{S}} K d\mathcal{A} = 2\pi\chi$
  where χ is the *Euler number* of the triangulation. (thm. 11.5, Gauss-Bonnet for compact surfaces [1]) Special cases of this theorem:
    - For the *unit-sphere* $\mathcal{S}^2$: $\chi = 2$ and $\int\int_{\mathcal{S}^2} K d\mathcal{A} = 4\pi$. (p. 264 [1])
    - For *any deformed sphere*: $\chi = 2$ and $\int\int_{\mathcal{S}^2} K d\mathcal{A} = 4\pi$. (p. 264-265 [1])
- The *Euler number* χ of a *triangulation* of a *compact surface* $\mathcal{S}$ *depends only on* $\mathcal{S}$ and *not* on the *choice of triangulation.* (cor. 11.2 [1])
- The *Euler number* of the compact surface T_g with *genus* g is $2-2g$. (thm. 11.6 [1])
- $\int\int_{T_g} K d\mathcal{A} = 4\pi(1-g)$. (cor. 11.3 [1])

- If a *triangulation* of a *compact surface* has Euler number χ, V vertices, E edges, F triangles, then:
  $3F = 2E, E = 3(V - \chi), V \geq \frac{1}{2}(7 + \sqrt{49 + 24\chi})$. (exc. 11.5 [1])
- For a *triangulation* of the *sphere* with n curvilinear triangles where r triangles meet at each vertex:
  - There are $3n/r$ vertices
  - There are $2n/2$ edges
  - $\frac{6}{r} - \frac{4}{n} = 1$
  - $r \leq 5$

  (exc. 11.6 [1])

10.0.29 For (non-unit speed) curves on surfaces:

$\overrightarrow{\gamma}(t)$.

- The *length* of a *curve* $\overrightarrow{\gamma}$ on an *arbitrary surface* can be computed by breaking $\overrightarrow{\gamma}$ into a *segment for each patch.* (p. 98 [1])
- κ_n and κ_g are those of $\overrightarrow{\gamma}$ *reparametrised* to *unit-speed.* (p. 128 [1])
- κ_1 and κ_2 *principal curvatures* of $\overrightarrow{\sigma}$ with non-zero principal vectors $\overrightarrow{t}_1$, $\overrightarrow{t}_2$. Then the *normal curvature* of $\overrightarrow{\gamma}$ is $\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$, where θ is the angle between $\overrightarrow{\gamma}'$ and $\overrightarrow{t}_1$. (cor. 6.1, Euler's Theorem [1])
- $\overrightarrow{\gamma}$ is a *line of curvature* $\Leftrightarrow$ $\overrightarrow{N}' = -\lambda\overrightarrow{\gamma}'$ for some scalar λ, which is then the *principal curvature.* (exc. 6.18, Rodrigues' Formula [1])
- $\overrightarrow{\gamma}$ is a *line of curvature* $\Leftrightarrow$ *geodesic curvature* k_g *vanishes* everywhere. (exc. 6.19 [1])
- Surfaces $\mathcal{S}_1, \mathcal{S}_2$ *intersect* at curve $\mathcal{C}$ which is *line of curvature* of $\mathcal{S}_1$. $\mathcal{C}$ is a *line of curvature* of $\mathcal{S}_2$ $\Leftrightarrow$ the *angle* between the *tangent planes* of $\mathcal{S}_1$ and $\mathcal{S}_2$ is *constant* along $\mathcal{C}$. (exc. 6.20 [1])
- Let $\overrightarrow{\sigma} : W \to \mathbb{R}^3$ be a *smooth function* defined on $W \underset{open}{\subseteq} \mathbb{R}^3$ such that for each *fixed value* of u, v or w repsectively, $\overrightarrow{\sigma}(u, v, w)$ is a *regular surface patch.* Assume that $\overrightarrow{\sigma}_u.\overrightarrow{\sigma}_v = 0, \overrightarrow{\sigma}_v.\overrightarrow{\sigma}_w = 0, \overrightarrow{\sigma}_w.\overrightarrow{\sigma}_u = 0$. This means that the three families of surfaces is a *triply orthogonal system.* Then:
  - $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are *diagonal* for each of the surfaces in the triply orthogonal system.
  - The *intersection* of any surface from one of the families of the triply orthogonal system with any surface from another familiy is a *line of curvature* of both surfaces. (Dupin's Theorem)

  (exc. 6.21 [1])
- At every point of an *asymptotic curve on surface* $\mathcal{S}$: $K \leq 0$. (exc. 7.8 [1])
- The *torsion* τ of an *asymptotic curve* on $\mathcal{S}$ is related to the *gaussian* by: $\tau^2 = -K$. (exc. 7.10 [1])

- The following holds *along* $\overrightarrow{\gamma}$ in $\overrightarrow{\sigma}$: $\overrightarrow{N}'.\overrightarrow{N}' + 2H\overrightarrow{N}'.\overrightarrow{\gamma}' + K\overrightarrow{\gamma}'.\overrightarrow{\gamma}' = 0$. (exc. 7.10 [1])
- $\overrightarrow{\gamma}$ is a *curve* in *smooth surface* $\mathcal{S}$ and *some point* $\overrightarrow{\gamma}(t_0)$ of $\overrightarrow{\gamma}$ lies in a surface patch $\overrightarrow{\sigma}$ of $\mathcal{S}$, then $\overrightarrow{\gamma}(t) = (u(t), v(t))$ will hold for all t in some open interval containing t_0.
  There fore we may restrict ourselves to curves of this form. (p. 74 [1])
- Any *geodesic* has *constant speed.* (prop. 8.1 [1])
- We can always *assume geodesics* to be *unit-speed.* (p. 172 [1])
- A *curve* on a surface is a *geodesic* $\Leftrightarrow \forall t : \kappa_g(t) = 0$. (prop. 8.2 [1])
- The *torsion* of a *geodesic* with *nowhere vanishing curvature* is *equal* to it's *geodedic torsion.* (exc. 8.4 [1])
- If a *geodesic* on $\mathbb{S}$ lies in a *plane* and has *nowhere vanishing curvature* then $\overrightarrow{\gamma}$ is a *line of curvature* of $\mathbb{S}$. (exc. 8.5 [1])
- $\overrightarrow{\gamma}$ on $\mathcal{S}$ is a *geodesic* $\Leftrightarrow$ for *any part of* $\overrightarrow{\gamma}(t) = \overrightarrow{\sigma}(u(t), v(t))$ of $\overrightarrow{\gamma}$ contained in a surface patch $\overrightarrow{\gamma}$ of $\mathcal{S}$, the following two equations (called the *geodesic equations*) are *satisfied*:
  - $\frac{d}{dt}(Eu' + Fv') = \frac{1}{2}(E_u(u')^2 2F_u u'v' + G_u(v')^2)$
  - $\frac{d}{dt}(Fu' + Gv') = \frac{1}{2}(E_v(u')^2 2F_v u'v' + G_v(v')^2)$

  These are ofthen difficult or impossible to solve. (thm. 8.1 [1])
- The *parameter* of any *curve satisfying* the *geodesic equations* is *proportional* to it's *arc-length.* (exc. 8.12 [1])
- Let P be a *point* on $\mathcal{S}$, and let $\overrightarrow{t}$ be a *unit-tangent vector* to $\mathcal{S}$ at P. Then there *exists* a *unique unit-speed geodesic* $\overrightarrow{\gamma}$ on $\mathcal{S}$ which *passes through* P and has *tangent* vector $\overrightarrow{t}$ there. (cor. 8.1 [1])
- An *isometry* between two surfaces takes the *geodesics* of *one surface* to the *geodesics* of the *other surface.* (cor. 8.2 [1])
- *Geodesics* on $\mathcal{S}$ where $\mathcal{S} \underset{closed}{\subseteq} \mathbb{R}^3$ can be *extended indefinately.* (p. 196 [1])

10.0.30 For *regular* curves on surfaces:

$\overrightarrow{\gamma}(t)$.

- $\kappa_n = \frac{L(u')^2 + 2Mu'v' + N(v')^2}{E(u')^2 + 2Fu'v' + G(v')^2}$. (exc. 6.16 [1])

10.0.31 For *smooth* curves on surfaces:

$\overrightarrow{\gamma}(t)$.

- *Smooth curve* $\overrightarrow{\gamma} :]\alpha, \beta[\to \mathbb{R}^3$ *contained in* the image of a *surface patch* $\overrightarrow{\sigma} : U \to \mathbb{R}^3$ *contained in* the atlas of the *smooth surface* $\mathcal{S} \Rightarrow$
  there *exists* a map $]\alpha, \beta[\to U$, say $t \mapsto (u(t), v(t)))$ such that $\overrightarrow{\gamma}(t) = \overrightarrow{\sigma}(u(t), v(t))$.
  u, v *smooth* $\Leftrightarrow \overrightarrow{\gamma}$ is a *curve in* $\mathcal{S}$. (p. 74 [1])

11 Curve on Surface Results

- $\kappa_n = 2\sqrt{5}$ for the *circle* $\overrightarrow{\gamma} = (\cos t, \sin t, 1)$ on *elliptic paraboloid* $\overrightarrow{\sigma}(u, v) = (u, v, u^2 + v^2)$. (exc. 6.5 [1])
- $\kappa_g = \pm\frac{1}{r}\sin\theta = \pm\frac{1}{R}\tan\theta$ for any *circle* on a *sphere*, where r is radius of circle, R is radius of sphere and the circle is parallel to latitude θ. (exc. 6.8 [1])
- For *surface of revolution* $\overrightarrow{\sigma}(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$
  where $u \mapsto (f(u), 0, g(u))$ is a unit-speed curve in $\mathbb{R}^3$:
  - For a *meridian*, $v =$ constant: $\kappa_g = 0$
  - For a *parallel*, $u =$ constant: $\kappa_g = \frac{f'}{f}$

  (exc. 6.9 [1])
- *Meridians* and *parallels* of a *surface of revolution* are *lines of curvature.* (exc. 6.18 [1])
- For *ruled surface* $\overrightarrow{\sigma}(u, v) = \overrightarrow{\gamma}(u) + v\delta(u)$:
  - If $\overrightarrow{\delta}$ is the *principal normal* $\overrightarrow{n}$ of $\overrightarrow{\gamma}$ or it's *binormal* $\overrightarrow{b}$, then $K = 0 \Leftrightarrow \overrightarrow{\gamma}$ is *planar.* (exc. 7.4 [1])
  - If $\overrightarrow{\gamma}$ is a *curve on a surface* $\mathcal{S}$ and $\overrightarrow{\delta}$ is the *unit normal* of $\mathcal{S}$, then $K = 0 \Leftrightarrow \overrightarrow{\gamma}$ is a *line of curvature* of $\mathcal{S}$. (exc. 7.4 [1])
- For the *tube* around a *unit-speed* curve $\overrightarrow{\gamma} : U \to \mathbb{R}^3$:
  $\overrightarrow{\sigma}(s, \theta) = \overrightarrow{\gamma}(s) + a(\cos\theta\,\overrightarrow{n}(s) + \sin\theta\,\overrightarrow{b}(s))$.
  The *parameter curves* of the tube obtained by fixing s are *cicular geodesics* on $\mathcal{S}$. (ex. 8.2 [1])
- For the *ellipsiod* $\mathcal{S}$ $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$, a curve $\overrightarrow{\gamma}(t) = (f(t), g(t), h(t)$ is a geodesic if and only if $(f'', g'', h'') = \lambda(\frac{f}{p^2}, \frac{g}{q^2}, \frac{h}{r^2})$ for some scalar $\lambda(t)$. The *curvature* of $\overrightarrow{\gamma}$ is $S(t)/R(t)^2$, where $S(t)$ is the distance from the centre of the ellipsoid to the tangent plane of $\mathcal{S}$ at $\overrightarrow{\gamma}(t)$ and $2R(t)$ is the length of the diameter of the ellipsiod parallel to $\overrightarrow{\gamma}'(t)$. (exc. 8.3 [1])
- The following are *always geodesics*
  (i.e. have parametrisations which are geodesics):
  - Any (part of a) *straight line* on a surface (prop. 8.3 [1])
  - The *rulings* of any *ruled surface* (e.g.: *generators* of a *generalised cylinder* or a *generalised cone*, the *straight lines* of a *hyperboloid* of *one sheet*). (ex. 8.1 [1])
  - Any *normal section* of a surface (prop. 8.4 [1])
  - The *intersection* of a *generalised cylinder* with a *plane perpendicular* to the *rulings* of the cylinder. (ex. 8.3 [1])
- The following are *all possible geodesics* for the following surfaces:
  - *Sphere*: The *great circles* (ex. 8.2, 8.4, 8.6 [1])

  - *Plane*: The *straight lines* (ex. 8.1, 8.5 [1])
  - *Circular unit cylinder*: Always *either* of the form:
    $\vec{\gamma}(t) = (\cos t, \sin t, mt + c)$ (i.e. *circular helixes* with radius 1 and pitch $2\pi|m|$ - which are *circles* if $m = 0$) *or*:
    are *lines parallel* to the *z-axis* (ex. 8.7 [1])
  - *Circular cone*: Isometric to the plane - see exc. 8.7 [1]
  - *Generalised cylinder*: The *constant speed* curves whose *tangent* make a *constant angle* with the *rulings* (exc. 8.8 [1])
  - *Hyperboloid of one sheet* obtained by rotating the hyperbola curve $x^2 - z^2 = 1$, $x > 0$ around z-axis:
    - The *unit-circle* in the *xy-plane*
    - The *meridians*
    - $0 < \Omega < 1$: The geodesics run through $z \in]-\infty, +\infty[$ in spirals around the z-axis
    - $\Omega > 1$: The geodesics run from $-\infty$ through some $z < 0$ and towards $+\infty$ in an arc. Correspondingly it may also run from $+\infty$ through some $z > 0$ and towards $-\infty$ in an arc.
    - $\Omega = 1$: The geodesics run in spirals from $\pm\infty$ towards $z = 0$ (i.e. the unit-circle in the xy-plane) and getting arbitrarily close to $z = 0$ without every reaching it.

    (ex. 8.9 [1])

- On the *surface of revolution* $\vec{\sigma}(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$, where we assume $f > 0$ and $(\frac{df}{du})^2 + (\frac{dg}{du})^2 = 1$:
  - *Every meridian* is a geodesic.
  - A *parallel* $u = u_0$ (say) is a *geodesic* $\Leftrightarrow$ $\frac{df}{du} = 0$ when $u = u_0$, i.e. u_0 is a *stationary point* of f.

  (prop. 8.5 [1])

- Let $\vec{\gamma}$ be a *geodesic* on a *surface of revolution* $\mathcal{S}$, let ρ be the *distance* of a *point* of $\mathcal{S}$ from the *axis of rotation*, and let ψ be the *angle* between $\vec{\gamma}'$ and the *meridians* of $\mathbb{S}$. Then $\rho \sin\psi$ is *constant* along $\vec{\gamma}$.
  Conversely, if $\rho\sin\psi$ is *constant* along some curve $\vec{\gamma}$ in $\mathcal{S}$, and if *no part* of $\vec{\gamma}$ is part of some *parallel* of $\mathcal{S}$, then $\vec{\gamma}$ is a *geodesic*. (prop. 8.6, Clairaut's Theorem [1])

- *Pseudosphere geodesics*: *Images under* $\vec{\vec{\sigma}}(u, w) = (\frac{1}{w}\cos v, \frac{1}{w}\sin v, \sqrt{1 - \frac{1}{w^2}} - \cosh^{-1} w)$ of the *parts* of the *circles* in the *uv-plane* given by $(v - v_0)^2 + w^2 = \frac{1}{\Omega^2}$ and lying in $w > 1$ are geodesics.
  In this case $\vec{\vec{\sigma}}(u, w)$ is a reparametrisation of $\vec{\sigma}(u, v) = (e^u \cos v, e^u \sin v, \sqrt{1 - e^{2u}} - \cosh^{-1}(e^{-u}))$ where $w = e^{-u}$.
  The geodesics can be described as the *meridians* and some geodesics which *can't be continued indefinately.* (exc. 8.8 [1])

- The *geodesics* on the *pseudosphere* in the *disc model* are described in exc. 8.17 [1]

- The *geodesics* on the *spheroid* and the *torus* are described in exc. 8.18 [1]
- The *circular half cone* is *isometric* to the *'sector'* of the *plane* with angle $2\sqrt{\pi}$. The *geodesics* on a circular half cone correspond to *possibly broken line segments* in the plane.
  The following holds for geodesics in the half cone (if the conditions mentioned in excercise 8.19 in [1] are met):
  - There *exists* a *geodesic* passing *through any 2 points* (however it is *not unique*)
  - Any two *distinct geodesics may intersect* in *more than one point*
  - There *exists geodesics* which do *not intersect each other*
  - *Any geodesic* can be *continued indefinately*
  - A *geodesic* defines the *shortest distance* between *any two of it's points*

  (exc. 8.19 [1])
- Let $\overrightarrow{\gamma}(\phi)$ be the *equator* of the *unit sphere* S^2 with the *longitude* ϕ as parameter, and let $\overrightarrow{\gamma}^{\phi}(\theta)$ be the *meridian* parametrised by *latitude* θ and passing through the point on the equator with longitude ϕ. This is the usual latitude longitude patch whose first fundamental form is $d\theta^2 + \cos^2\theta d\phi^2$ (ex. 8.10 [1])
- The *circumference* C_R of the *geodesic circle* with *centre* P and radius R (on the *geodesic polar patch???*) is $C_R = 2\pi R(1 - \frac{K(P)}{6}R^2 + remainder)$ where $remainder/R^2$ tends to zero as R tends to zero. The *area* A_R inside this geodesic circle is $A_R = \pi R^2(1 - \frac{K(P)}{12}R^2 + remainder)$ where *remainder* tends to zero as R tends to zero. (exc. 10.3 [1])
- Let ABC be a triangle on the *geodesic polar patch* $\overrightarrow{\sigma}$ whose *sides are arcs* of *geodesics* where A is a point on $r = 0$ in the patch. Let AB be the parameter curve $\theta = 0$, AC be the parameter curve $\theta = \angle A$ and BC be parametrised by $\overrightarrow{\gamma}(\theta) = \overrightarrow{\sigma}(f(\theta), \theta)$ for some smooth function f and $0 \leq \theta \leq \angle A$. Then:
  - If $\lambda = \|\overrightarrow{\gamma}'\|$ then $f'' - \frac{f'\lambda'}{\lambda^2} = \frac{1}{2}\frac{\partial G}{\partial r}$
  - If $\psi(\theta)$ is the *angle* between $\overrightarrow{\sigma}_r$ and the *tangent vector* to BC at $\overrightarrow{\gamma}(\theta)$ then $\psi'(\theta) = -\frac{\partial\sqrt{G}}{\partial r}(f(\theta), \theta)$
  - $\int\int_{ABC} K d\mathcal{A}_{\overrightarrow{\sigma}} = \angle A + \angle B + \angle C - \pi$

  (exc. 10.4 [1])
- The *area* of the *lune* of a unit-sphere - i.e. the area between 2 great circles $= 2\theta$, where θ is the angle between the planes containing the great circles $=$ the angle between the circles at the pole points. (ex. 5.8 [1])
- The *unit-sphere convered by triangles* of *great circle arcs.* $F =$ number of triangles, $E =$ number of edges. $V =$ number of vertices. Then: $3F = 2E, 2V - F = 4, V - E + F = 2$. (exc. 5.21 [1])

12 Properties of Well-Known Surfaces

Using (θ, r) means polar coordinates and (θ, ϕ) means spherical coordinates.

Name	$\vec{\sigma}$ followed by some properties	Ref.
Plane	$\vec{a} + u\vec{p} + v\vec{q}$ for $\vec{p}, \vec{q}$ *perpendicular unit vectors*	(ex. 5.1 [1])
	$E = 1, F = 0, G = 1$. First fundamental form: $du^2 + dv^2$.	
	$L = M = N = 0$.	(ex. 6.1 [1])
	$\kappa_1 = \kappa_2 = 0$, and all points are *planar*	(ex. 6.5 [1])
Unit sphere	$(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$	(ex. 5.2 [1])
	$\kappa_1 = \kappa_2 = 1$, and all points are *umbilic.*	(ex. 6.3, 6.5 [1])
	Any tangent is a principal vector	
	$E = 1, F = 0, G = \cos^2\theta$	
	$L = 1, M = 0, N = \cos^2\theta$	(ex. 6.2 [1])
	$K = H = 1$	(ex. 7.1 [1])
	Also: $\mathcal{S}^2 = \{\vec{v} \in \mathbb{R}^3 \mid \|\vec{v}\| = 1\}$	(p. 166 [1])
	Area $\mathcal{A}(\mathcal{S}) = 4\pi$.	(exc. 6.16 [1])
Helicoid	$(v\cos u, v\sin u, \lambda u)$ surface swept out by an	(exc. 4.14 [1])
	aeroplane propeller - both moving at constant speed	
	$\kappa_1 = \frac{\lambda}{\sqrt{\lambda^2+v^2}}$, $\kappa_2 = -\frac{\lambda}{\sqrt{\lambda^2+v^2}}$	(exc. 6.15 [1])
	$E = (\lambda^2 + v^2), F = 0, G = 1$	
	$L = 0, M = \frac{\lambda}{\sqrt{\lambda^2+v^2}}, N = 0$	
	$K = -\frac{\lambda^2}{(\lambda^2+v^2)^2}$ (FIXME: Why squared denom.?)	(exc. 7.2 [1])
Catenoid	$(\cosh u\cos v, \cosh u\sin v, u)$ for $U \in \mathbb{R}$ and	(exc. 4.18 [1])
	$-\pi < v < \pi$ or $0 < v < 2\pi$.	
	Obtained by rotating $x = \cosh z$ around z-axis.	
	$\kappa_1 = sech^2 u, \kappa_2 = -sech^2 u$	(exc. 6.15 [1])
	$E = \cosh^2 u, F = 0, G = \cosh^2 u$	
	$L = -1, M = 0, N = 1$	
	$K = -sech^4 u$	(exc. 7.2 [1])
Hyperbolic -paraboloid	$(r\cosh\theta, r\sinh\theta, r^2)$ parametrisation of the part of	(exc. 4.8 [1])
	the equation: $z = x^2 - y^2$ where $z > 0$	
Elliptic -paraboloid	$(u, v, u^2 + v^2)$	(exc. 6.1 [1])
	$L = 2\lambda, M = 0, N = 2\lambda$.	
Ellipsiod	Level surface: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ where a, b, c constants $\neq 0$	(exc. 4.9 [1])
	smooth	
Ruled -surface	$\vec{\gamma}(u) + v\overrightarrow{\delta(u)}$ The union of straight lines called the	(ex. 4.12, 7.3 [1])
	rulings of the surface. $\vec{\gamma}$ is a curve that meets all	
	these lines and $\vec{\delta}$ is a non-zero direction vector	
	in the direction of the line passing through $\vec{\gamma}(u)$.	
	$\vec{\sigma}$ is *regular* if $\vec{\gamma}' + v\vec{\delta}'$ and $\vec{\delta}$ are linearly independent.	
	So to get a surface $\vec{\gamma}$ must never be tangent to the rulings.	
	$\vec{\sigma}_u = \vec{\gamma}' + v\vec{\delta}'; \vec{\sigma}_v = \vec{\delta}; \vec{N} = (\vec{\sigma}_u \times \vec{\sigma}_u)/\|\vec{\sigma}_u \times \vec{\sigma}_u\|$	
	$K = \frac{-(\vec{\delta}'.\vec{N})^2}{EG-F^2}$ which is ≤ 0	
	$M = \vec{\delta}'.\vec{N}, N = 0$	
	$\vec{\sigma}$ *conformal* $\Leftrightarrow$ $\vec{\delta}(u)$ *independent* of u and	(exc. 5.13 [1])
	$\vec{\gamma}$ lies in a *plane perpendicular* to $\vec{\delta}$.	

Name	$\overrightarrow{\sigma}$ followed by some properties	Ref.
Pseudo-sphere	$(f(u)\cos v, f(u)\sin v, g(u))$ where	(p. 151-154 [1])
	$f(u) = e^{2u}, g(u) = \sqrt{1-e^{2u}} - \cosh^{-1}(e^{-u})$	
	It is a *surface of revolution* with the *tractrix* profile curve:	
	$z = \sqrt{1-x^2} - \cosh^{-1}(\frac{1}{x})$	
	Distance from P on *tractrix* to	
	intersection of *tangent line* at P with *z-axis* is *constant* 1	
	The *parallels* are circles of *radius* e^u with *length* $2\pi e^u$	(exc. 7.11 [1])
	Total *area* is 2π and *all points* are *hyperbolic*	(exc. 7.11 [1])
	$\kappa_1 = -(e^{-2u}-1)^{-1/2}, \kappa_2 = (e^{-2u}-1)^{1/2}$	(exc. 7.11 [1])
	Setting $w = e^{-u}$ gives the *upper half-plane model*	(exc. 7.12 [1])
	reparametrisation $\overrightarrow{\sigma}_1(v,w)$ with first fundamental form:	(exc. 7.12 [1])
	$E = G = \frac{1}{w^2}, F = 0$	(exc. 7.12 [1])
	Setting $U = \frac{v^2+w^2-1}{v^2+(w+1)^2}, V = \frac{-2v}{v^2+(w+1)^2}$ for $U^2+V^2 < 1$	(exc. 7.12 [1])
	gives the *disc model* reparametrisation $\overrightarrow{\sigma}_2(U,V)$	(exc. 7.12 [1])
	with first fundamental form:	(exc. 7.12 [1])
	$E = G = \frac{1}{(1-U^2-V^2)^2}, F = 0$	(exc. 7.12 [1])
Móbius-band	$\overrightarrow{\sigma}(t,\theta) = ((1-t\sin\frac{\theta}{2})\cos\theta, (1-t\sin\frac{\theta}{2})\sin\theta, t\cos\frac{\theta}{2})$	(ex. 4.9 [1])
	where $U = \{(t,\theta) \in \mathbb{R}^2 \mid -\frac{1}{2} < t < \frac{1}{2}, 0 < \theta < 2\pi\}$	
	and $\widetilde{U} = \{(t,\theta) \in \mathbb{R}^2 \mid -\frac{1}{2} < t < \frac{1}{2}, -\pi < \theta < \pi\}$	
	smooth non-orientable surface consisting of *regular patches*	
Generalised-cylinder	$\overrightarrow{\sigma}(u,v) = \overrightarrow{\gamma}(u) + v\overrightarrow{a}$ where $U = \{(u,v) \in \mathbb{R}^2 \mid \alpha < a < \beta\}$	(ex. 4.10 [1])
	surface obtained by *translating* curve $\overrightarrow{\gamma} :]\alpha,\beta[\to \mathbb{R}^3$	
	in the *direction* of the *unit-vector* $\overrightarrow{a}$	
	$\overrightarrow{\sigma}$ *smooth.* $\overrightarrow{\sigma}$ *regular* $\Leftrightarrow$ $\overrightarrow{\gamma}'$ *never parallel* to $\overrightarrow{a}$	
	$\overrightarrow{\sigma}_u = \overrightarrow{\gamma}'$, $\overrightarrow{\sigma}_v = \overrightarrow{a}$	
	$E = 1, F = 0, G = 1$	(ex. 5.3 [1])
	Any generalised cylinder is *isometric* to part of a plane.	(exc. 5.7 [1])
	Simpler parametrisation for $\overrightarrow{a} = (0,0,1)$:	(ex. 4.10 [1])
	$\overrightarrow{\sigma}(u,v) = (f(u), g(u), v)$	
Circular-cylinder	$(\cos v, \sin v, u)$ defined on:	(ex. 4.10 [1])
	$\{(u,v) \in \mathbb{R}^2 \mid 0 < u < 2\pi\}$ and $\{(u,v) \in \mathbb{R}^2 \mid -\pi < u < \pi\}$	
	$\kappa_1 = 1, \kappa_2 = 0$, and all points *parabolic*	(ex. 6.4, 6.5 [1])
	Principal vector $\overrightarrow{t}_1$ is a multiple of $(-\sin v, \cos v, 0)$	
	Principal vector $\overrightarrow{t}_2$ is a multiple of $(0,0,1)$	
	$E = 1, F = 0, G = 1$	
	$L = 0, M = 0, N = 1$	(ex. 6.2 [1])
	$K = 0, H = \frac{1}{2}$	(ex. 7.1 [1])
Generalised-cone	$\overrightarrow{\sigma}(u,v) = (1-v)\overrightarrow{p} + v\overrightarrow{\gamma}(u)$ the union of straight	(ex.4.11 [1])
	lines passing through a fixed point $\overrightarrow{p}$ and	
	the points of a curve $\overrightarrow{\gamma} :]\alpha,\beta[\to \mathbb{R}^3$	
	$\overrightarrow{\sigma}$ *smooth.* $\overrightarrow{\sigma}$ *regular* if $v \neq 0$ and none of the	
	straight lines forming the cone are tangent to $\overrightarrow{\gamma}$	
	Any generalised cone is *isometric* to part of a plane.	(exc. 5.7 [1])
	Simpler parametrisation when $\forall u : \Vert\overrightarrow{\gamma}(u)\Vert = 1$	(ex. 5.4 [1])
	and $\overrightarrow{\gamma}$ unit-speed: $\overrightarrow{\sigma}(u,v) = v\overrightarrow{\gamma}(u)$	
	$E = v^2, F = 0, G = 1.$	
	Simpler representation if $\overrightarrow{p} = (0,0,0)$	(ex.4.11 [1])
	and $\overrightarrow{\gamma}$ in $z = 1$ plane: $\overrightarrow{\sigma}(u,v) = v(f(u), g(u), 1)$	

Name	$\overrightarrow{\sigma}$ followed by some properties	Ref.
Surface of	$\overrightarrow{\sigma}(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ Rotation of a	(ex. 4.13 [1])
-revolution	plane curve (called profile curve, assumed to be in xz-plane)	
	around a straight line (assumed to be the z-axis) in the plane.	
	Profile: $\overrightarrow{\gamma}(u) = (f(u), 0, g(u))$	
	The *parallels* are the circles from rotating at each point	
	of the profile curve.	
	The *meridians* are copies of the profile at fixed angles.	
	$\overrightarrow{\sigma}_u = (f'\cos v, f'\sin v, g')$, $\overrightarrow{\sigma}_v = (-f\sin v, f\cos v, 0)$	
	$\overrightarrow{\sigma}_u \times \overrightarrow{\sigma}_v$ non-vanishing if $\forall u : f(u) \neq 0$ and $\overrightarrow{\gamma}$ is regular.	
	$\overrightarrow{\sigma}$ *smooth* if: v restricted to an open interval of length $\leq 2\pi$	
	and $\overrightarrow{\gamma}$ does not self-intersect and $\forall u : f(u) > 0$.	
	Area $\mathcal{A}(\mathcal{S}) = 2\pi \int g(u)du$.	(exc. 5.16 [1])
	If $\forall u : f(u) > 0$ and profile $u \mapsto (f(u), 0, g(u))$ *unit-speed*:	(ex. 6.2 [1])
	$E = 1, F = 0, G = f^2, L = f'g'' - f''g', M = 0, N = fg'$.	
Stereographic	$(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1})$ *conformal* parametrisation	(p. 110 [1])
-projection	of the *unit-sphere* except north pole.	
Cylinder	$(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z)$ (a.k.a. *Lambert projection*)	(thm. 5.4 [1])
-projection	An *equiareal diffeomorphism* (Archimedes' Theorem).	
Mercator's	(*sech* $u \cos v$, *sech* $u \sin v$, $\tanh u$)	(exc. 4.19 [1])
-projection	A *regular* surface patch for the *unit-sphere.*	
	Meridians and parallels correspond to	
	perpendicular straight lines in the plane under $\overrightarrow{\sigma}$.	
	Conformal.	(exc. 5.11 [1])
Loxodrome	A *curve* on the *unit-sphere* that *intersect* the *meridians*	(exc. 4.20 [1])
	at a *fixed angle* α.	
	Loxodromes correspond to straight lines	
	in the uv-plane under $\overrightarrow{\sigma}$.	
	A *unit-speed* loxodrome in Mercator's surface patch satisfies:	
	$u' = \cos\alpha \cosh u$, $v' = \pm \sin\alpha \cosh u$	
Quadric	$(\sinh u \sinh v, \sinh u \cosh v, \sinh u)$	(exc. 5.1 [1])
- cone	$E = 2\cosh^2 u, F = 2\sinh u \cosh u, G = \sinh^2 u$	
Paraboloid	$(u - v, u + v, u^2 + v^2)$	(exc. 5.1 [1])
-of revolut.	$E = 2 + 4u^2, F = 4uv, G = 2 + 4v^2$	
Hyperbolic	$(\cosh u, \sinh u, v)$	(exc. 5.1 [1])
-cylinder	$E = \cosh^2 u + \sinh^2 u, F = 0, G = 1$	
Paraboloid	$(u, v, u^2 + v^2)$	(exc. 5.1 [1])
-of revolut.	$E = 1 + 4u^2, F = 4uv, G = 1 + 4v^2$	
Tangent	$\overrightarrow{\gamma}(u) + v\overrightarrow{\gamma}'(u)$ where $\overrightarrow{\gamma}$ is *unit-speed*	(p. 103-104 [1])
-developable	the union of the tangent lines to the curve $\overrightarrow{\gamma}$ in $\mathbb{R}^3$.	
	$\kappa = \lVert \overrightarrow{\gamma}'' \rVert > 0$ and $v \neq 0 \Rightarrow \overrightarrow{\sigma}$ *regular.*	
	So we must exclude $\overrightarrow{\gamma}$ from surface for regularity.	
	$E = 1 + v^2\kappa^2, F = G = 1$.	
	Any tangent developable is *isometric* to part of a plane.	(prop. 5.1 [1])

Name	$\overrightarrow{\sigma}$ followed by some properties	Ref.
Tube	$\overrightarrow{\sigma}(s,\theta) = \overrightarrow{\gamma}(s) + a(\overrightarrow{n}(s)\cos\theta + \overrightarrow{b}(s)\sin\theta)$	(exc. 5.17 [1])
-around $\overrightarrow{\gamma}$	Radius a. $\overrightarrow{n}$ principal normal, $\overrightarrow{b}$ binormal.	
	Area in $s_0 < s < s_1, 0 < \theta < 2\pi$ where s_0, s_1 constant is:	
	$2\pi a(s_1 - s_0)$.	
Torus	$((a + b\cos\theta)\cos\phi, (a + b\cos\theta)\sin\phi, b\sin\theta)$ where	(exc. 4.10 [1])
	$(\theta, \phi) \in U_i \underset{open}{\subseteq} \mathbb{R}^2$ for suitable U_i.	
	$\kappa_1 = \frac{1}{b}, \kappa_2 = \frac{\cos\theta}{a+b\cos\theta}$	
	The torus is also the level surface:	
	$(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$	
	$\int\int K d\mathcal{A}_{\overrightarrow{\sigma}} = 0$	(exc. 7.5 [1])
	Area is $4\pi^2 ab$.	(exc. 5.16 [1])
Right	$(v\cos(\theta(u)), v\sin(\theta(u)), u)$	(exc. 4.21 [1])
- conoid	Ruled surface with rulings parallel to a given plane	
	Π (assumed to be xy-plane) and pass a given straight line	
	$\mathcal{L}$ (assumed to be z-axis) perpendicular to Π.	
	$\overrightarrow{\sigma}$ *regular* surface patch where $\theta(u)$ is the angle	
	that the ruling through $(0, 0, u)$ makes with the z-axis.	
	$\theta(u)$ smooth function of u. $\theta(u) = u$ gives a helicoid.	
???	$z = f(x, y)$ where f *smooth function*	(exc. 7.3 [1])
	$K = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2}$	

References

[1] Andrew Pressley. *Elemetary Differential Geometry*, Springer Ferlag 2001.