Geometric Curves Summary

Ánoq of the Sun, Hardcore Processing *

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1 Vectors, Functions, Conventions etc.

Let $\overrightarrow{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$:

- *Differentiation*: $\overrightarrow{\gamma}'(t) = \frac{d\overrightarrow{\gamma}}{dt} = (\frac{d\gamma_1}{dt}, \dots, \frac{d\gamma_n}{dt})$ (p. 4 [1])
- *Dot product differentiation of vector functions*: $\frac{d}{dt}(\overrightarrow{a} \cdot \overrightarrow{b}) = \frac{d\overrightarrow{a}}{dt} \cdot \overrightarrow{b} + \overrightarrow{a} \cdot \frac{d\overrightarrow{b}}{dt}$ (p. 9 [1])
- *Cross product differentiation of vector functions*: $\frac{d}{dt}(\overrightarrow{a} \times \overrightarrow{b}) = \frac{d\overrightarrow{a}}{dt} \times \overrightarrow{b} + \overrightarrow{a} \times \frac{d\overrightarrow{b}}{dt}$ (p. 37 [1])
- *Resdistribution of cross product and dot product*: $b \cdot (a \times c) = -c \cdot (a \times b)$ (p. 38 [1])
- *Chain rule differentiation*: $\frac{d}{dx} f(g(x)) = \frac{df}{dy}\frac{dg}{dx}$ for $y = (g(x))$ (p. 11 [1], p. 67 [2]) $f(g(x)) = f'(g(x))g'(x)$ (p. 56 [2])
- *Length* of vector $\overrightarrow{v} = (v_1, \dots, v_n) \underset{def}{=} \|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$ (p. 7 [1])
- $\overrightarrow{\gamma}(t)$ *smooth* $\underset{def}{=} \gamma_1, \gamma_2, \dots, \gamma_n$ smooth $\underset{def}{=}$ for $i \in \{1, 2, \dots, n\} : \frac{d\gamma_i}{dt}, \frac{d^2\gamma_i}{dt^2}, \dots$ exists (p. 4 [1])
- *Cross triple product identity*: $\forall a, b, c \in \mathbb{R}^3 : a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ (p. 25 [1])
- *For 3 righthanded orthonormal vectors*: $a = b \times c, c = a \times b, b = c \times a$ (p. 36-37 [1])
- *Rigid Motion in* $\mathbb{R}^2$: A map $M : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $M = T_{\overrightarrow{a}} \circ R_\theta$ where R_θ is an anti-clockwise rotation by an angle θ about the origin: $R_\theta(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$ and $T_{\overrightarrow{a}}$ is the translation by the vector $\overrightarrow{a}$: $T_{\overrightarrow{a}}(\overrightarrow{v}) = \overrightarrow{v} + \overrightarrow{a}$ for any vectors $\overrightarrow{(x, y)}, \overrightarrow{v} \in \mathbb{R}^2$. (p. 30 [1])
- *Fresnel's integrals*: $\int_0^t \cos(\frac{j^2}{2})dj$ and $\int_0^t \sin(\frac{j^2}{2})dj$ (ex. 2.3 [1])

1.1 Conventions

- We assume all *parametrised* curves to be *smooth* in [1]. (p. 4 [1])
- All *parametrized* curves in [1] will be assumed to be *regular* from page 47 (Chapter 5)
- All *simple closed* curves in [1] will be assumed to be *positively oriented* from p. 49

2 Curve Representations

2.1 Representations

- *Level curves* given by *cartesian equations* or *implicit formulas*: $f(x_1, \dots, x_n) = c$.
  Curve is the set of points $\mathcal{C} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid f(x_1, \dots, x_n) = c\}$.
  We can also have multiple functions at the same time (p. 1-2 [1])

- *Parametrised curve* in $\mathbb{R}^n \underset{def}{=}$ a map $\overrightarrow{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^n$ where $\infty \leq \alpha < \beta \leq \infty$ (def. 1.1 [1])

- *Unit-speed curve* $\underset{def}{=}$ parametrised curve $\overrightarrow{\gamma}(t)$ where $\forall t \in]\alpha, \beta[: \|\overrightarrow{\gamma}'(t)\| = 1$ (def. 1.4 [1])

2.2 More definitions

- *parametrization of (part of)* $\mathcal{C} \underset{def}{=}$ A parametrised curve whose image is contained in a level curve $\mathcal{C}$ (p. 2 [1])

2.3 Some Theorems

- Let $f(x, y)$ be a smooth function of 2 variables and let a level curve be $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$. Assume that $\forall p \in \mathcal{C} : (\frac{\partial f}{\partial x}(p) \neq 0) \vee (\frac{\partial f}{\partial y}(p) \neq 0)$. Then $P = (x_0, y_0) \in \mathcal{C} \Rightarrow$ there is a regular parametrised curve $\overrightarrow{\gamma}(t)$ defined on an open interval containing 0, such that $\overrightarrow{\gamma}$ passes through P when $t = 0$ and $\forall t : \overrightarrow{\gamma(t)} \in \mathcal{C}$ (thm. 1.1 [1]) (justified p. 17 [1] by the 2 variable form of Taylor's theorem and the inverse function theorem) (FIXME: Generalisation in exc. 4.16 [1])

- If $f(x, y)$ satisfies the above it even holds that if $\mathcal{C}$ given by $f(x, y) = 0$ is *connected* there is a a regular parametrised curve $\overrightarrow{\gamma}(t)$ whose image is *the whole* of $\mathcal{C}$. More over if $\mathcal{C}$ does not close up, $\overrightarrow{\gamma}$ can be made injective. If $\mathcal{C}$ does close up then $\overrightarrow{\gamma}$ maps some closed interval $[\alpha, \beta]$ *onto* $\mathcal{C}$, $\overrightarrow{\gamma}(\alpha) = \overrightarrow{\gamma}(\beta)$ and $\overrightarrow{\gamma}$ is *injective* on $]\alpha, \beta[$ (p. 18-19 [1])

- Let $\overrightarrow{\gamma}$ be a regular parametrised curve, and let $\overrightarrow{\gamma}(t_0) = (x_0, y_0)$ be a point in the image of $\overrightarrow{\gamma}$. Then there is a smooth real-valued function $f(x, y)$, defined for x and y in open intervals containing x_0 and y_0 respectively, and satisfying the conditions in theorem 1.1, such that $\overrightarrow{\gamma}(t)$ is contained in the level curve $f(x, y) = 0$ for all values of t in some open interval containting t_0 (thm 1.2 [1]) (note: it is not always possible to find a single $f(x, y)$ for the entire $\overrightarrow{\gamma}$ because $\overrightarrow{\gamma}$ may be selfintersecting). Also holds in $\mathbb{R}^n$ (exc. 1.18 [1])

2.4 Conversion between representations

- *Implicit to parametrized*: Find $\overrightarrow{\gamma}(t) = (\gamma_1(t), \dots, \gamma_n(t))$ such that $f(\gamma_1(t), \dots, \gamma_n(t)) = c$ and $\overrightarrow{\gamma}$ passes through all of $\mathcal{C}$ (p. 2-3 [1])

- *Parametrized to level*: Find $f(x_1, \dots, x_n)$ such that $f(\overrightarrow{\gamma}(t)) = c$ for all t in parameter range and $f(x_1, \dots, x_n) \neq c$ for any other $(x_1, \dots, x_n)$ (p. 3 [1])

- The parametrised curve $\overrightarrow{\widetilde{\gamma}} :]\widetilde{\alpha}, \widetilde{\beta}[\to \mathbb{R}^n$ is a *reparametrisation* of $\overrightarrow{\gamma} :]\alpha, \beta[\to \mathbb{R}^n \underset{def}{=}$ there is a smooth bijective map $\phi :]\widetilde{\alpha}, \widetilde{\beta}[\to]\alpha, \beta[$ (the *reparametrisation map*) whose inverse map $\phi^{-1} :]\alpha, \beta[\to]\widetilde{\alpha}, \widetilde{\beta}[$ is also smooth where $\forall \widetilde{t} \in]\widetilde{\alpha}, \widetilde{\beta}[: \overrightarrow{\widetilde{\gamma}}(\widetilde{t}) = \overrightarrow{\gamma}(\phi(\widetilde{t}))$ (def. 1.5 [1])

3 Some Well-known Named Curves

Using (θ, r) means polar coordinates.

Name	Implicit	Parametrized ($\overrightarrow{\gamma}(t) =$)	Param Rg	Ref.
Parabola	$x^2 - y = 0$	(t, t^2)	$\mathbb{R}$	(p. 3 / ex. 1.8 [1])
Circle	$x^2 + y^2 - r^2 = 0$	$(r\cos(t), r\sin(t))$	$]-\epsilon, 2\pi[$	(p. 3 [1])
Hyperbola	$x^2 - y^2 = 1$			(p. 19 [1])
Astroid	$x^{2/3} + y^{2/3} = 1$	$(\cos^3(t), \sin^3(t))$	$\mathbb{R}$	(p. 3 [1])
Cycloid		$a(t - \sin t, 1 - \cos t)$	$\mathbb{R}$	(exc. 1.7 [1])
Viviani's Curve		$(\cos^2 t - \frac{1}{2}, \sin t \cos t, \sin t)$		(exc. 1.9 [1])
Logarithmic Spiral		$(e^t \cos t, e^t \sin t)$		(exc. 1.10 [1])
Catenary		$(t, \cosh t)$		(exc. 1.11 [1])
Twisted Cubic		(t, t^2, t^3)	$\mathbb{R}$	(ex. 1.7 [1])
Cissoid of Diocles	$\sin\theta \tan\theta = r$	$(t^2, \frac{t^3}{\sqrt{1-t^2}})$	$]-1, 1[$	(exc. 1.15 [1])
Limaçon	(self-intersect)	$(1 + 2\cos t)(\cos t, \sin t)$		(p. 20 [1])
Cicular Helix		$(a\cos\theta, a\sin\theta, b\theta)$	$\mathbb{R}$	(ex. 2.1 [1])
Cornu's Spiral		$(\int_0^t \cos(\frac{j^2}{2})dj, \int_0^t \sin(\frac{j^2}{2})dj)$	$\mathbb{R}$	(ex. 2.3 [1])

Unit-speed parametrisations.

Name	Parametrized ($\overrightarrow{\gamma}(t) =$)	Param Rg	$\overrightarrow{\gamma}'(t)$	κ_s	Ref.
Circle	$(r\cos(\frac{t}{r}), r\sin(\frac{t}{r}))$	$]-\epsilon, 2\pi[$	$(-\sin(\frac{t}{r}), \cos(\frac{t}{r}))$	$\frac{1}{r}$	(p. 32 [1])

4 Local Curve Properties

4.0.1 For parametrised curves

$\overrightarrow{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^n$

- *Tangent vector* $\underset{def}{=} \overrightarrow{t} = \overrightarrow{\gamma}'(t) = \frac{d\overrightarrow{\gamma}}{dt} = \lim\limits_{\delta t \to 0} \frac{\overrightarrow{\gamma}(t+\delta t) - \overrightarrow{\gamma}(t)}{\delta t}$ (def. 1.2 [1])
- *Speed* at $\gamma(t) \underset{def}{=} \|\overrightarrow{\gamma}'(t)\|$ (def. 1.4 [1])
- $\overrightarrow{\gamma}(t)$ is a *regular point* $\underset{def}{=} \overrightarrow{\gamma}'(t) \neq 0$ (def. 1.6 [1])
- $\overrightarrow{\gamma}(t)$ is a *singular point* $\underset{def}{=} \overrightarrow{\gamma}'(t) = 0$ (def. 1.6 [1])

4.0.2 For *regular* parametrised curves

$\overrightarrow{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^n$

- *Curvature* $\kappa(s)$ in $\mathbb{R}^3$ at any *regular point* $\overrightarrow{\gamma}(s) \underset{def}{=} \frac{\|\overrightarrow{\gamma}''(s) \times \overrightarrow{\gamma}'(s)\|}{\|\overrightarrow{\gamma}'(s)\|^3}$ (prop. 2.1 [1])

4.0.3 For *unit speed* parametrised curves

$\overrightarrow{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^n$

- *Unit tangent vector* $\underset{def}{=} \overrightarrow{t} = \overrightarrow{\gamma}'(t)$ (p. 28 [1])
- *Curvature* $\kappa(s)$ at the point $\overrightarrow{\gamma}(s) \underset{def}{=} \|\overrightarrow{\gamma}''(s)\| = \|\overrightarrow{t}'(s)\|$ (def 2.1 [1])

4.0.4 For *unit speed* parametrised *plane* curves

$\overrightarrow{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^2$

- *Signed unit normal* $\underset{def}{=} \overrightarrow{n}_s$ given as $\overrightarrow{t}$ rotated $\frac{\pi}{2}$ anti-clockwise (p. 28 [1])
- *Signed curvature* $\kappa_s(s) \underset{def}{=} \kappa_s$ such that $\overrightarrow{\gamma}'' = \kappa_s \overrightarrow{n}_s$ (p. 28 [1])

4.0.5 For parametrised *plane curvilinear polygons*

- A *curvilinear polygon* in $\mathbb{R}^2$ $\underset{def}{=}$ a *continuous map* $\overrightarrow{\pi} : \mathbb{R} \to \mathbb{R}^2$ such that for some real number a and some points (called *vertices*) $0 = t_0 < t_1 < t_2 < \cdots < t_n = a$ the following holds:
  - $\overrightarrow{\pi}(t) = \overrightarrow{\pi}(t') \Leftrightarrow t' - t$ is an *integer multiple* of a
  - $\overrightarrow{\pi}$ is *smooth* on *each* of the *open interavals* (called *edges*) $]t_0, t_1[$, $]t_1, t_2[, \ldots,]t_{n-1}, t_n[$
  - The *one-sided derivatives* $(\overrightarrow{\pi}')^-(t_i) = \lim\limits_{t \nearrow t_i} \frac{\overrightarrow{\pi}(t) - \overrightarrow{\pi}(t_i)}{t - t_i}$, $(\overrightarrow{\pi}')^+(t_i) = \lim\limits_{t \searrow t_i} \frac{\overrightarrow{\pi}(t) - \overrightarrow{\pi}(t_i)}{t - t_i}$ *exist* for $i = 1, \ldots, n$ and are *non-zero* and *non-parallel*

  (def. 11.2 [1])
- A curvilinear polyon is *positively oriented* $\underset{def}{=}$ for all t where $\overrightarrow{\pi}(t)$ is *not a vertex*, the vector $\overrightarrow{n}_s$ obtained by *rotating* $\overrightarrow{\pi}'$ *anti-clockwise* by $\pi/2$ should point *into* $int(\overrightarrow{\pi})$. (p. 252 [1])
- The *Jordan Curve Theorem* applies to curvilinear polygons. (p. 252 [1])

4.0.6 For *unit speed* parametrised *space* curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^3$

- *Principal normal vector* $\vec{n}(s)$ at $\vec{\gamma}(s)$ is defined by: $\vec{n}(s) = \frac{1}{\kappa(s)} \vec{t}\,'(s)$ if the *curvature* $\kappa(s)$ *is non-zero* (p. 36 [1])
- $\vec{n}(s)$ is a unit vector (since $\kappa = \| \vec{t}\,' \|$), perpendicular to $\vec{t}$ (p. 36 [1])
- *Binormal vector* $\vec{b}(s)$ at $\vec{\gamma}(s)$ is defined by: $\vec{b}(s) = \vec{t}(s) \times \vec{n}(s)$ (p. 36 [1])
- *Torsion* τ is defined by the equation $\vec{b}\,' = -\tau \vec{n}$ if the *curvature* $\kappa(s)$ *is non-zero* (p. 37 [1])

4.0.7 For *regular* parametrised *space* curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^3$

- *Torsion* τ is defined as the torsion of a unit-speed reparametrisation of $\vec{\gamma}$ if the *curvature* $\kappa(s)$ *is non-zero* (p. 37 [1])
- *Torsion* τ is defined by $\tau = \frac{(\vec{\gamma}\,' \times \vec{\gamma}\,'') \cdot \vec{\gamma}\,'''}{\| \vec{\gamma}\,' \times \vec{\gamma}\,'' \|^2}$ if the *curvature* $\kappa(s)$ *is non-zero* (prop 2.3 [1])
- If the curvature $\kappa(s)$ *is non-zero* we have the *Frenet-Serret equations*:
$$\begin{array}{rcccc} \vec{t}\,' & = & & \kappa \vec{n} & \\ \vec{n}\,' & = & -\kappa \vec{t} & & \tau \vec{b} \\ \vec{b}\,' & = & & -\tau \vec{n} & \end{array}$$
(thm. 2.2 [1])

5 Local Curve Properties for Well-Known Named Curves

Name	$\vec{\gamma}\,'(t)$	$\|\vec{\gamma}\,'(t)\|$	arc-length	Ref.
Log. Spiral	$e^t(\cos t - \sin t, \sin t + \cos t)$	$\sqrt{2e^{2t}}$	$\sqrt{2}(e^t - 1)$	(ex. 1.4 / 1.6 [1])
Twisted C.	$(1, 2t, 3t^2)$	$\sqrt{1 + 4t^2 + 9t^4}$	(ex. 1.7 [1])	(ex. 1.7 [1])
Circ. Helix	$(-a \sin\theta, a\cos\theta, b)$	$\sqrt{a^2 + b^2}$		(ex. 2.1 [1])

Name	$\vec{\gamma}\,''(s)$	$\|\vec{\gamma}\,''(s)\|$	$\kappa(s)$	Ref.
Circle		$\frac{1}{r}$		(p. 24 [1])
Circ. Helix	$(-a\cos\theta, -a\sin\theta, 0)$		$\frac{\|a\|}{a^2+b^2}$	(ex. 2.1 [1])

Name	$\vec{\gamma}\,'''(s)$	$\|\vec{\gamma}\,'''(s)\|$	$\tau(s)$	Ref.
Circ. Helix	$(a\sin\theta, -a\cos\theta, 0)$		$\frac{b}{a^2+b^2}$	(ex. 2.4 [1])

6 Local Curve Theorems

6.0.8 For *unit speed* parametrised curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^n$

- $\vec{\gamma}''(t)$ is $\vec{0}$ or perpendicular to the tangent vector $\vec{\gamma}'(t)$ (prop 1.2 [1])

6.0.9 For parametrised *plane* curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^2$

- Applying a reflection in a straight line to a plane curve changes the sign of it's signed curvature (exc. 2.12 [1])

6.0.10 For *unit speed* parametrised *plane* curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^2$

- $\kappa(s) = |\kappa_s(s)|$ (p. 28 [1])
- Let $\vec{\phi}(s)$ be the angle through which some fixed unit vector $\vec{v}$ must be rotated anti-clockwise to bring it to conincide with the unit tangent vector $\vec{t}$ of $\vec{\gamma}$. Then: $\kappa_s = \frac{d\phi}{ds}$ (prop. 2.2 [1])
- Let $k :]\alpha, \beta[\to \mathbb{R}$ be any smooth function. Then there is a unit-speed curve $\vec{\gamma} :]\alpha, \beta[\to \mathbb{R}^2$ whose signed curvature is k. Further, if $\vec{\tilde{\gamma}} :]\alpha, \beta[\to \mathbb{R}^2$ is any other unit-speed curve whose signed curvature is k, then there is a rigid motion M of $\mathbb{R}^2$ such that $\forall s \in]\alpha, \beta[: \vec{\tilde{\gamma}}(s) = M(\vec{\gamma}(s))$ (thm. 2.1 [1])
- $\vec{n}'_s = -\kappa_s \vec{t}$ (exc. 2.3 [1])
- If $\kappa \neq 0$ for all t we define the *centre of curvature* $\vec{\epsilon}(s)$ of $\vec{\gamma}$ at the point $\vec{\gamma}(s)$ to be: $\vec{\epsilon}(s) = \vec{\gamma}(s) + \frac{1}{\kappa_s(s)} \vec{n}_s(s)$. (exc. 2.8 [1])
- The circle with centre $\vec{\epsilon}$ and radius $|\frac{1}{\kappa_s(s)}|$ is called the *osculating circle* which is tangent to $\vec{\gamma}$ at $\vec{\gamma}(s)$ and has the same curvature as $\vec{\gamma}$ at that point. (exc. 2.8 [1])
- If $\vec{\epsilon}$ is regarded as the parametrisation of a new curve, it is called the *evolute* of $\vec{\gamma}$. (exc. 2.9 [1])
- If $\vec{\iota}(s) = \vec{\gamma}(s) + (l - s)\vec{\gamma}'(s)$ is regarded as the parametrisation of a new curve for some l, it is called the *involute* of $\vec{\gamma}$. (exc. 2.10 [1])

6.0.11 For *regular* parametrised *plane* curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^2$

- κ_s is a smooth function of t (exc. 2.4 [1])
- For a constant λ the *parallel curve* $\vec{\gamma}^{\lambda}(t)$ is defined by: $\vec{\gamma}^{\lambda}(t) = \vec{\gamma}(t) + \lambda \vec{n}_s(t)$. If $|\lambda \kappa_s(t)| < 1$ for all t, then $\vec{\gamma}^{\lambda}(t)$ is regular and it's signed curvature is $\kappa_s/(1 - \lambda \kappa_s)$ (exc. 2.7 [1])

- The *evolute* of $\overrightarrow{\gamma}$ is defined to be the evolute of the unit-speed reparametrisation of $\overrightarrow{\gamma}$. (exc. 2.9 [1])
- The *involute* of $\overrightarrow{\gamma}$ is defined to be the involute of the unit-speed reparametrisation of $\overrightarrow{\gamma}$. (exc. 2.10 [1])
- The *involute of the evolute* of $\overrightarrow{\gamma}$ is a parallel curve of $\overrightarrow{\gamma}$ (exc. 2.11 [1])
- The *evolute of the involute* of $\overrightarrow{\gamma}$ is $\overrightarrow{\gamma}$ (exc. 2.11 [1])

6.0.12 For *unit speed* parametrised *space* curves

$\overrightarrow{\gamma}(t) :]\alpha, \beta[\rightarrow \mathbb{R}^3$

- $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ is an orthonormal basis (FIXME: Called Frenet's frame?) and $\overrightarrow{b} = \overrightarrow{t} \times \overrightarrow{n}, \overrightarrow{n} = \overrightarrow{b} \times \overrightarrow{t}, \overrightarrow{t} = \overrightarrow{n} \times \overrightarrow{b}$ (p. 36-37 [1])
- Let $\overrightarrow{\tilde{\gamma}}(s)$ and $\overrightarrow{\gamma}$ be 2 unit-speed curves in $\mathbb{R}^3$ with the *same curvature* $\kappa(s) > 0$ and the *same torsion* $\tau(s)$ for all s. Then there is a *rigid motion* M of $\mathbb{R}^3$ such that $\overrightarrow{\tilde{\gamma}} = M(\overrightarrow{\gamma})$ for all s. Further, if k and t are smooth functions with $k > 0$ everywhere, there *exists* a unit-speed curve in $\mathbb{R}^3$ whose curvature is k and whose torsion is t. (thm. 2.3 [1])

7 Global Pointwise Local Curve Properties

7.0.13 For parametrised curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^n$:

- *Arc-length* from $t_0 \underset{def}{=} s(t) = \int_{t_0}^{t} \|\vec{\gamma}'(u)\| du$ (def. 1.3 [1])
- $\vec{\gamma}$ *regular* $\underset{def}{=}$ all points are regular ($\forall t \in]\alpha, \beta[: \vec{\gamma}'(t) \neq 0$) (def. 1.6 [1])

8 Global Pointwise Local Curve Theorems

8.0.14 For parametrised curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^n$:

- *Tangent* of a parametrised curve *constant* $\Rightarrow$ the image of the curve is (part of) a *straight line* (a point if tangent is $\vec{0}$) (prop. 1.1 [1])

8.0.15 For *regular* parametrised curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^n$

- Any *reparametrisation* of a *regular* curve is *regular* (prop. 1.3 [1])
- A parametrised curve $\vec{\gamma}(t)$ has a *unit-speed reparametrisation* (and $\phi = s^{-1}$ is the reparametrisation map) $\Leftrightarrow$ $\vec{\gamma}(t)$ is *regular* (prop. 1.5 [1])
- Let $\vec{\gamma}$ be a *regular curve* and $\vec{\tilde{\gamma}}$ be a *unit speed reparametrisation* of $\vec{\gamma}$: $\forall t$: $\vec{\tilde{\gamma}}(u(t)) = \vec{\gamma}(t)$ where u is a smooth function of t. Let s be the arc-length of $\vec{\gamma}$ starting at any point. Then $u = \pm s + c$ where c is a constant. Conversely, if u is given as $u = \pm s + c$ for some c and with either sign, then $\vec{\tilde{\gamma}}$ is a unit-speed reparametrisation of $\vec{\gamma}$. (cor. 1.1 [1])
- $\vec{\gamma}(t)$ *regular* $\Rightarrow$ it's *arc-length* starting at any point is a *smooth* function of t (prop. 1.4 [1])
- *Torision* τ is a *smooth function* of t whenever it is defined. (exc. 2.18)

8.0.16 For *regular* parametrised *plane* curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^2$

- *Curvature* of a parametrised regular curve is a *positive constant* $\Rightarrow$ the image of the curve is (part of) a *circle* (ex. 2.2 [1])

8.0.17 For *unit speed* parametrised *space* curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^3$

- $\vec{b}'(s)$ is perpendicular to both $\vec{b}(s)$ and $\vec{t}(s)$ (p. 37 [1])
- *Constant* (non-zero) *curvature* $\kappa(s)$ and *zero torsion* τ $\Rightarrow$ $\vec{\gamma}$ is (part of) a *circle* with radius $\frac{1}{\kappa}$ (prop. 2.5 [1])

- $\kappa(t) > 0$ and $\tau(t) \neq 0$ for all t.
  $\vec{\gamma}$ lies on *the surface of a sphere* $\Leftrightarrow \frac{\tau}{\kappa} = \frac{d}{ds}(\frac{\kappa'}{\tau\kappa^2})$ (exc. 2.21 [1])

- *Curvature* $\kappa(t) \neq 0$ for all $t \Rightarrow \vec{\delta}(t) = \frac{d\vec{\gamma}(t)}{dt}$ is *regular*. Further, if s is an arc-length parameter for $\vec{\delta}$ then $\kappa = \frac{ds}{dt}$. Curvature κ' of $\vec{\delta}$: $\kappa' = \sqrt{(1 + \frac{\tau^2}{\kappa^2})}$ (exc. 2.19 [1])

8.0.18 For *regular* parametrised *space* curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^3$

- If curvature is nowhere 0 (so that the torsion τ of $\vec{\gamma}$ is defined). Then the image of $\vec{\gamma}$ is contained in a plane (which is perpendicular to $\vec{b}$) if and only if τ is zero at every point of the curve. (prop. 2.4 [1])

- Curves with *constant curvature* $\kappa > 0$ and *constant torsion* τ are *circular helixes*. (exc. 2.17 [1])

- *General helix* $\underset{def}{=}$ regular curve where the *tangent vector* $\vec{t}$ makes a *fixed angle* θ with a *fixed unit vector* $\vec{a}$. Then $\tau = \pm\kappa \cot\theta$ (exc. 2.20 [1])

- $\tau = \lambda\kappa$ (where λ constant) $\Rightarrow \vec{\gamma}$ is a *general helix*. (exc. 2.20 [1])

- Any *circular helix* is also a *general helix*. (exc. 2.20 [1])

9 Global Curve Properties

9.0.19 For *regular* parametrised *plane* curves

$\vec{\gamma}(t) :]\alpha, \beta[\to \mathbb{R}^2$:

- A *simple closed* curve in $\mathbb{R}^2$ with period $a \in \mathbb{R}, a > 0 \underset{def}{=}$ a regular curve $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^2$ such that $\vec{\gamma}(t) = \vec{\gamma}(t') \Leftrightarrow t' - t = ka, k \in \mathbb{Z}$ (def. 3.1 [1])

9.0.20 For *simple closed* parametrised *plane* curves with period a

$\vec{\gamma}(t) : \mathbb{R} \to \mathbb{R}^2$:

- *Length* of $\vec{\gamma} \underset{def}{=} l(\vec{\gamma}) = \int_0^a \|\vec{\gamma}'(t)\| dt$ (p. 49 [1])
- *Positively oriented* $\underset{def}{=}$ signed unit normal $\vec{n}_s$ points *into* $int(\vec{\gamma})$ for all t. Corresponds to *counter clockwise* traversal of a circular curve in the plane. Positive orientation can always be achieved by replacing t with $-t$ if necessary. (p. 49-50 [1])
- *Area* contained by $\vec{\gamma} \underset{def}{=} \mathcal{A}(int(\vec{\gamma})) = \int\int_{int(\vec{\gamma})} dxdy$ (p. 50 [1])
- $\vec{\gamma}$ has an *interior* $int(\vec{\gamma})$ and *exterior* $ext(\vec{\gamma})$ satisfying the following:
  1. $int(\vec{\gamma})$ is *bounded* (i.e. contained within a sufficiently large circle)
  2. $ext(\vec{\gamma})$ is *unbounded*
  3. $int(\vec{\gamma})$ and $ext(\vec{\gamma})$ are both *connected* regions (i.e. any 2 points in the region can be joined by a curve contained entirely in the region). However, joining $int(\vec{\gamma})$ with $ext(\vec{\gamma})$ will cross $\vec{\gamma}$. (Jordan's Curve Theorem, p. 48 [1])
- $\vec{\gamma}$ *convex* $\underset{def}{=} int(\vec{\gamma})$ convex - i.e. the straight line between any 2 points in $int(\vec{\gamma})$ is contained entirely in $int(\vec{\gamma})$. (p. 55 [1])
- A *vertex* of $\vec{\gamma} \underset{def}{=}$ a point where $\vec{\gamma}$'s *signed curvature* κ_s has a *stationary point*, i.e. where $\frac{d\kappa_s}{dt} = 0$.
  This definition is *independent of parametrisation* of $\vec{\gamma}$. (def. 3.2 [1])

10 Global Curve Theorems

10.0.21 For *simple closed* parametrised *plane* curves with period a

$\vec{\gamma}(t) : \mathbb{R} \to \mathbb{R}^2$:

- We can *always assume* that a simple closed curve is *unit-speed* with *period equal* to it's *length*. (p. 49 [1])
- $l(\vec{\gamma})$ and $\mathcal{A}(int(\vec{\gamma}))$ are *unchanged* when applying a rigid motion to $\vec{\gamma}$. (exc. 3.1 [1])
- Period $a \Rightarrow$ tangent vector $\vec{t}(t + a) = \vec{t}(t)$, signed unit normal $\vec{n}_s(t + a) = \vec{n}_s(t)$, signed curvature $\kappa_s(t + a) = \kappa_s(t)$. (exc. 3.4 [1])

- $\mathcal{A}(int(\vec{\gamma})) \leq \frac{1}{4\pi} l(\vec{\gamma})^2$, with equality if and only if $\vec{\gamma}$ is a circle. (Isoperimetric Equality, thm. 3.1 [1])

- $\vec{\gamma}$ has *at least 4 vertices.* (4 Vertex Thm., thm. 3.2 [1])

10.0.22 For *simple closed positively oriented* parametrised *plane* curves with period a

$\vec{\gamma}(t) : \mathbb{R} \to \mathbb{R}^2$:

- $\vec{\gamma} = (x(t), y(t))$ has period $a \Rightarrow \mathcal{A}(int(\vec{\gamma})) = \frac{1}{2}\int_0^a (xy' - yx')dt$. Unchanged if $\vec{\gamma}$ is reparametrized. (prop. 3.1 [1])

11 Well-Known Named Simple Closed Curves

Name	$\vec{\gamma}(t)$	property	value	Ref.
Circle	$(\cos(\frac{2\pi t}{a}), \sin(\frac{2\pi t}{a}))$	$int(\vec{\gamma})$	$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$	(ex. 3.1 [1])
		$ext(\vec{\gamma})$	$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 > 1\}$	
Ellipse	$(a\cos t, b\sin t)$	$\mathcal{A}(int(\vec{\gamma}))$	πab	(exc. 3.2 [1])

11.0.23 Not Simple Closed

Name	$\vec{\gamma}(t)$	property	Ref.
Limaçon	$((1 + 2\cos t)\cos t, (1 + 2\cos t)\sin t)$	$\forall t : \vec{\gamma}(t + 2\pi) = \vec{\gamma}(t)$	(exc. 3.3 [1])

12 Misc Curve Equations

Using (θ, r) means polar coordinates.

Name	equation	followed by some properties	Ref.
Circle	$r = A\sin(\theta - \alpha)$	diameter $= A$	(p. 53-54 [1])
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}dt \geq 2\pi\sqrt{ab}$	(exc. 3.5 [1])
	- with equality if and only if $a = b$ (i.e. for a circle)		
Tractrix	$z = \sqrt{1 - x^2} - \cosh^{-1}(\frac{1}{x})$		(p. 151-154 [1])
	Distance from any point P on tractrix to *intersection* of *tangent line* at P with *z-axis* is 1		

13 Misc Parametrised Curves

Name	$\vec{\gamma}(t)$	followed by some properties	Ref.
Ellipse	$(a\cos t, b\sin t)$ where $a, b > 0$	$\kappa_s(t) = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$	(ex. 3.2 [1])
	A *simple closed, convex* curve with period 2π. Vertices at $t \in \{0, \pi/2, \pi, 3\pi/2\}$		

14 Misc Curve Results

- The *limaçon* has only 2 vertices (exc. 3.8 [1])

15 Misc Theorems

- $\vec{n}(t)$ *unit vector* (i.e. $\|\vec{n}(t)\| = 1$) as a *smooth function of* $t \Rightarrow$ the dot product $\vec{n}'(t) \cdot \vec{n}(t) = 0$ for all t (i.e. $\vec{n}'(t)$ is $\vec{0}$ or perpendicular to $\vec{n}(t)$) (prop 1.2 [1])

- Let (a_{ij}) be a *skew-symmetric* 3×3 *matrix* (i.e. $\forall i, j : a_{ij} = a_{ji}$). Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be smooth functions of s satisfying the differential equations $\vec{v}_i' = \sum_{j=1}^{3} a_{ij} \vec{v}_j$ for $i = 1, 2, 3$ and suppose that for some s_0, $\vec{v}_1(s_0), \vec{v}_2(s_0), \vec{v}_3(s_0)$ are *orthonormal*. Then $\vec{v}_1(s), \vec{v}_2(s), \vec{v}_3(s)$ are *orthonormal for all* s. (exc. 2.22 [1])

- For all *smooth functions* $f(x,y), g(x,y)$ and $\vec{\gamma}$ *simple closed*, *positively oriented*: $\int\int_{int(\vec{\gamma})} (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) dx dy = \int_{\vec{\gamma}} f(x,y)dx + g(x,y)dy$. (Green's Theorem, p. 50 [1])

- Let $F : [0, \pi] \to \mathbb{R}$ be a *smooth function* such that $F(0) = F(\pi) = 0$. Then $\int_0^{\pi} (\frac{dF}{dt})^2 dt \geq \int_0^{\pi} F(t)^2 dt$ with equality holding if and only if $\forall t \in [0, \pi] : F(t) = A \sin t$ where A is a constant. (Wirtinger's Inequality, prop. 3.2 [1])

References

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