Differential Equations Summary

Ánoq of the Sun, Hardcore Processing *

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1 Differential Equations

1. order differential equation system: $\frac{dx}{dt} = f(t,x)$ where $I \subseteq$ (interval) $\mathbb{R}, f : I \times \mathbb{R}^k \to \mathbb{R}^k$. Also written as: $\frac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots .x_k), \ldots, \frac{dx_k}{dt} = f_k(t, x_1, x_2, \ldots .x_k)$ where $f = (f_1, f_2, \ldots, f_k)$ (p. 7.1 [1])

- Let $J \underset{subinterval}{\subseteq} I, \varphi : J \to \mathbb{R}^k$. φ is a *solution* for $\frac{dx}{dt} = f(t,x) \doteq$ φ differentiable and $\forall t \in J : \varphi'(t) = f(t, \varphi(t))$ (def. 7.1 [1])

- φ is a *maximal solution* for $\frac{dx}{dt} = f(t,x) \doteq$ there is no solution on J' where $J \subset J'$ which equals φ on J (p. 7.1 [1])

- Any solution for $\frac{dx}{dt} = f(t,x)$ is continuous differentiable (p. 7.1 [1])

- Continuity of f is *not* enough to ensure a unique solution. E.g. $f(t,x) = \begin{cases} 2\sqrt{x} & t > t_0 \\ 0 & t \leq 0 \end{cases}$ has both the solution $\varphi(t) = 0$ and $\varphi(t) = \begin{cases} (t-t_0)^2 & t > t_0 \\ 0 & t \leq t_0 \end{cases}$ (ex. 7.2 [1])

- $\forall I \underset{interval}{\subseteq} \mathbb{R} : \forall \varphi = (\varphi_1, \ldots, \varphi_k) \in C(I, \mathbb{R}^k) : \forall [a,b] \in I :$ $|| \int_a^b \varphi(s) ds || \leq \int_a^b ||\varphi(s)|| ds$ for any norm on $\mathbb{R}^k$ (p. 7.2 [1])

- Let $(t_0, x_0) \in I \times \mathbb{R}^k, \varphi \in C(I, \mathbb{R}^k)$. $\forall t \in I : T\varphi(t) = x_0 + \int_{t_0}^t f(s, \varphi(s)) ds$ is the base function for $f(s, \varphi(s))$ with value x_0 for $t = t_0$ (main theorem of differential and integrational maths).
  This equation defines a map $T : C(I, \mathbb{R}^k) \to C(I, \mathbb{R}^k)$ (also called $T(t_0, x_0, I)$).
  That T has a fixpoint means that it solves the integral equation:
  $\varphi(t) = x_0 + \int_{t_0}^t f(s, \varphi(s)) ds$ for $t \in I$ (p. 7.2 - 7.3)

- The differential equation $\frac{dx}{dt} = f(t,x)$ with start value $x(t_0) = x_0$ has a solution $\varphi : I \to \mathbb{R}^k \Leftrightarrow \varphi \in C(I, \mathbb{R}^k)$ is fixpoint for $T(t_0, x_0, I)$ (notice that we require φ to be continuous - not differentiable) (thm. 7.3 [1])

- For a compact interval $K, C(K, \mathbb{R}^k)$ is complete with $||\varphi||_u = \sup_{t \in K} ||\varphi(t)||_\infty = \sup_{t \in K} \max(|\varphi_1(t)|, \ldots, |\varphi_k(t)|)$ (p. 7.3 [1], thm. 5.8 [1])

- K compact interval and $f : K \times \mathbb{R}^k \to \mathbb{R}^k$ meets a global Lipschitz condition: $\exists c > 0 : \forall t \in K : \forall x, y \in \mathbb{R}^k : ||f(t,x) - f(t,y)||_\infty \leq c||x-y||_\infty$ then $T = T(t_0, x_0, K)$ meets the Lipschitz condition: $\forall \varphi, \psi \in C(K, \mathbb{R}^k) : ||T\varphi - T\psi||_u \leq c \cdot l(K) \cdot ||\varphi - \psi||_u$ where $l(K)$ is the length of the interval K (lemma 7.4 [1])

- The global Lipschitz condition means that $\forall t \in K : f(t, \cdot) : \mathbb{R}^k \to \mathbb{R}^k$ meets a Lipschitz condition on all of $\mathbb{R}^k$ with the same c (rem. 7.5 [1])

1.1 Banach's Fixpoint Theorem

- Let $(x_n)_{n \geq 0}$ be a sequence in a metric space (M, d).
$\sum_{j=0}^{\infty} d(x_j, x_{j+1}) < \infty \Rightarrow (x_n)$ Cauchy (lemma 7.6 [1])

- For a set M the function $T : M \to M$ has a *fixpoint* $a \in M \underset{def}{=} T(a) = a$ (p. 7.5 [1])

- For a set M a fixpoint for T can sometimes (but not always!) be found be the *method of successive aproximations*: Chose some $x_0 \in M$. $x_1 = T(x_0), x_2 = T(x_1) = T^{\circ 2}(x_0), x_n = T(x_{n-1}) = T^{\circ n}(x_0), \ldots$. If $(x_n)_{n \geq 0}$ has a limit $a = \lim_{n \to \infty} x_n$ then a is a fixpoint for T since $T(a) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = a$ (p. 7.5 [1])

- Let (M, d) be a metric space. $T : M \to M$ is a *contraction of* $M \underset{def}{=}$ T meets a Lipschitz condition with *contraction constant* $0 < c < 1$
i.e.: $\forall x, y \in M : d(T(x), T(y)) \leq c \cdot d(x, y)$ (def. 7.7 [1])

- T a contraction of a complete metric space $(M, d) \Rightarrow T$ has exactly one fixpoint $a \in M$ and for any x_0 the sequence of iterated points $(T^{\circ n}(x_{n-1}))_{n \geq 1}$ will converge towards a (Banach's Fixpoint Theorem) (thm. 7.8 [1])

1.2 Existence and Uniqueness Theorems for 1. Order Differential Equation Systems

- Assume that $f : I \times \mathbb{R}^k \to \mathbb{R}^k$ meets the global Lipschitz condition on $K \times \mathbb{R}^k$ for all $K \underset{compact\ interval}{\subseteq} I$ (where the constant c may depend on K). For given $t_0 \in I, x_0 \in \mathbb{R}^k$ the differential equation system $\frac{dx}{dt} = f(t, x)$ has exactly one solution $\varphi : I \to \mathbb{R}^k$ which satisfies the start condition $\varphi(t_0) = x_0$ (thm. 7.9 [1])

- *Linear system of differential equations*:
$$\begin{aligned} \frac{dx_1}{dt} &= p_{11}(t)x_1 + \cdots + p_{1k}(t)x_k + q_1(t) \\ &\vdots \\ \frac{dx_k}{dt} &= p_{k1}(t)x_1 + \cdots + p_{kk}(t)x_k + q_k(t) \end{aligned}$$
With the colum vectors $x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}, q(t) = \begin{pmatrix} q_1(t) \\ \vdots \\ q_k(t) \end{pmatrix}$
and $k \times k$ matrices $P(t) = (p_{ij}(t))$ we can write the equation as:
$\frac{dx}{dt} = P(t)x + q(t)$ (p. 7.7 [1])

- $\forall (t_0, x_0) \in I \times \mathbb{R}^k$: There exists exactly one solution $\varphi : I \to \mathbb{R}^k$ for $\frac{dx}{dt} = P(t)x + q(t)$ where $\varphi(t_0) = x_0$ (thm. 7.10 [1])

- Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}^k$ be open and $f : \Omega \to \mathbb{R}^k$ a continuous function. f meets a *local Lipchitz condition* $\underset{def}{=} \forall (t_0, x_0) \in \Omega$ there exists a compact interval $K_0 = [t_0 - \epsilon, t_0 + \epsilon]$ and a compact box $D_0 = \{x \in \mathbb{R}^k \mid ||x - x_0||_\infty \leq r\}$ with $K_0 \times D_0 \subseteq \Omega$ and a constant $c > 0$ such that: $\forall t \in K_0 : \forall x, y \in D_0 : ||f(t,x) - f(t,y)||_\infty \leq c||x - y||_\infty$ (def. 7.11 [1])

- A solution for $\frac{dx}{dt} = f(t,x)$ with f as for the local Lipchitz condition is now $\varphi : J \to \mathbb{R}^k$ where $J \underset{open}{\subseteq} \mathbb{R}$ such that $\forall t \in J : (t, \varphi(t)) \in \Omega$ and $\forall t \in J : \varphi'(t) = f(t, \varphi(t))$ (p. 7.9 [1])

- Assume that $\Omega \underset{open}{\subseteq} \mathbb{R} \times \mathbb{R}^k$ and that $f : \Omega \to \mathbb{R}^k$ is continuous and meets a local Lipschitz condition then $\forall (t_0, x_0) \in \Omega$ there exists exactly one maximal solution $\varphi : J \to \mathbb{R}^k$ for the differential equation system $\frac{dx}{dt} = f(t,x)$ where $(t,x) \in \Omega$ and $\varphi(t_0) = x_0$. Any solution is a restriction of a maximal solution (thm. 7.12 [1])

- Assume that $\Omega \underset{open}{\subseteq} \mathbb{R} \times \mathbb{R}^k$, that $f = (f_1, \ldots, f_k) : \Omega \to \mathbb{R}^k$ is continuous and that all f_i in Ω has continuous partial derivatives $\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_k}$ after the last k variables. Then f meets a local Lipschitz condition (lemma 7.13 [1])

- $f(t,x) = 1 + x^2$ has for any $c \in \mathbb{R}$ the solution $\varphi(t) = tg(t - c), t \in]c - \frac{\pi}{2}, c + \frac{\pi}{2}[$. f meets a local Lipchitz condition and lemma 7.13 was used. $\varphi(t) \to \pm\infty$ for $t \to c \pm \frac{\pi}{2}$ so the solutions are maximal (ex. 7.14 [1])

1.3 Differential Equations of Higher Order

- Let $\Omega \underset{open}{\subseteq} \mathbb{R}^{n+1}$ and $F : \Omega \to \mathbb{R}$ be continuous. The solution for $\frac{d^n x}{dt^n} = F(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}})$ is an n times differentiable function $\varphi : J \to \mathbb{R}$ such that $\forall t \in J : (t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)) \in \Omega$ and $\forall t \in J : \varphi^{(n)}(t) = F(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t))$ (def. 7.15 [1])

- The solution $\varphi : J \to \mathbb{R}$ is *maximal* $\underset{def}{=}$ there does not exist any solution on an interval J' where $J \subset J'$ which equals φ on J (def. 7.15 [1])

- The solution $\varphi : J \to \mathbb{R}$ *passes trough* $(t_0, x_0, x_1, \ldots, x_{n-1}) \in \Omega \underset{def}{=}$ $t_0 \in J$ and $\varphi(t_0) = x_0, \varphi'(t_0) = x_1, \ldots, \varphi^{(n-1)}(t_0) = x_{n-1}$ (def. 7.15 [1])

- $\varphi : J \to \mathbb{R}$ is a solution for $\frac{d^n x}{dt^n} = F(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}})$ which passes through $(t_0, x_0, \ldots, x_{n-1})$ if and only if it is first coordinate function in a solution $\widetilde{\varphi} : t \mapsto (\varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t))$ for $t \in J$ with start condition $\widetilde{\varphi}(t_0) = (x_0, \ldots, x_{n-1})$ for the differential equation system:

$$
\begin{aligned}
\frac{dx_0}{dt} &= x_1 \\
\frac{dx_1}{dt} &= x_2 \\
&\vdots \\
\frac{dx_{n-2}}{dt} &= x_{n-1} \\
\frac{dx_{n-1}}{dt} &= F(t, x_0, x_1, \dots, x_{n-1})
\end{aligned}
$$

(p. 7.12-13 [1])

- Assume that the function $F : I \times \mathbb{R}^n \to \mathbb{R}$ is continuous and has continuous partial derivatives after the last n variables. Then there exists a unique maximal solution for $\frac{d^n x}{dt^n} = F(t, x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}})$ which passes through some given point $(t_0, x_0, \dots, x_{n-1}) \in I \times \mathbb{R}^n$ and any solution is a restriction of a maximal solution (thm. 7.16 [1])

References

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