Convergence of Sequences and Series Summary

Ánoq of the Sun, Hardcore Processing *

June 30, 2004

1 Sequences

1.1 About *sequences*

- ***Growing*** $\overset{def}{=}$ $a_1 \leq a_2 \leq \cdots$ (p. 212, [1])
- ***Falling*** $\overset{def}{=}$ $a_1 \geq a_2 \geq \cdots$ (p. 212, [1])
- ***Upwards limited*** $\overset{def}{=}$ $\exists M (\in \mathbb{R}?) : \forall n (\in I?) : a_n < M$ (p. 212, [1])
- ***Downwards limited*** $\overset{def}{=}$ $\exists M (\in \mathbb{R}?) : \forall n (\in I?) : a_n > M$ (p. 212, [1])

1.2 Convergence of Sequences

Premises: (x_n) **sequence of *real* numbers**

- (x_n) *upwards limited, growing* $\Rightarrow$ (x_n) *converges* (thm 8.1, [1]) (thm 1.5, [3])
- (x_n) *downwards limited, falling* $\Rightarrow$ (x_n) *converges* (thm 8.1, [1]) (thm 1.5, [3])
- A real sequence $(x_n)_{n=1}^{\infty}$ converges towards $a \in \mathbb{R} \Leftrightarrow$ $\liminf\limits_{n\to\infty} x_n = \limsup\limits_{n\to\infty} x_n = a$ (thm. 3.2 [3])

2 Series

2.1 Arithmetic Rules

- $\sum_{i=1}^{\infty}(a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$,
  provided that the 2 rightmost series *converge* (p. 215, [1])
- $\sum a_i$ *converges* $\Rightarrow \sum_{i=1}^{\infty} ca_i = c \sum_{i=1}^{\infty} a_i$,
  $\sum a_i$ *diverges* $\Rightarrow \sum ca_i$ *diverges* for $c \neq 0$. (p. 215, [1])

2.2 Convergence

- $\sum a_n$ *converges* $\Rightarrow \lim_{n\to\infty} a_n = 0$ (n'te led testen, p. 216, [1])
- If $f(x)$ is *positive*, *falling*, *continuous on* $[1, \infty[$ we have:
  $\sum_{n=1}^{\infty} f(n)$ *converges* $\Leftrightarrow \int_1^{\infty} f(x)dx$ *converges* (integral test, p. 216, [1])
- Assume $0 \leq a_n \leq b_n$ for all n. Then we have:
  $\sum b_n$ *converges* $\Rightarrow \sum a_n$ *converges*
  $\sum a_n$ *diverges* $\Rightarrow \sum b_n$ *diverges*
  (comparison test, p. 217, [1])

2.3 Known Series

- $\forall a, r \in \mathbb{R} : \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$.
  $|r| \geq 1 \Rightarrow$ *diverges* (geometric series thm. 8.2 p. 214, [1])
- The *p-series*: $p > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. Diverges for $p \leq 1$. Called *geometric series* for $p = 1$ (p. 217 [1])

References

[1] Arne Hole. *Klassisk Analyse og Lineær Algebra*, Universitetsforlaget 1998.

[2] Bergfinnur Durhuus. *Hilbert Rum med Anvendelser*, Matematisk Afdeling Københavns Universitet 1997.

[3] Mikael Rørdam (?). *Om de Reelle Tal - Supplerende Noter til 2AN*, Maths Department University of Copenhagen 2001.