Complex Analysis Summary

Ánoq of the Sun, Hardcore Processing *

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1 Misc Notation And Names

- For $G \underset{open}{\subseteq} \mathbb{C}$: if $f : G \to \mathbb{C}$ is written $w = f(z)$ we write: $f'(z) = \frac{dw}{dz} = \frac{df}{dz}$ for the *differential quotient* / *derivative* in $z \in G$ ⟨p. 9 ∈ [1]⟩
- A *stretching* function is called: DK: *homoteti*
- $K(a, r)$: the *ball* with *center* a and *radius* r. $K'(a, r) = K(a, r) \backslash \{a\}$

2 Areas in $\mathbb{C}$ (or $\mathbb{R}^2$)

- $G \subset \mathbb{C}$ **is an *area*** $\stackrel{def}{=}$
  G is *open* and any two points in G can be connected with a *stairline* in G (i.e. a line consisting only of vertical and horizontal line segments inside G) ⟨def. 1.8, p. 16 ∈ [1]⟩
- $G \subset \mathbb{C}$ **is ⟨ DK: ⟩ *enkeltsammenhængende*** $\stackrel{def}{=}$
  for 2 *arbitrary curves* γ_0, γ_1 in G with the *same starting points* a and the *same end points* b, we can define the *homotopy* function $H : [0, 1] \times [0, 1] \to G$, where:
  H is *continuous* and
  $\forall t \in [0, 1] : H(0, t) = \gamma_0(t), H(1, t) = \gamma_1(t)$ and
  $\forall s \in [0, 1] : H(s, 0) = a, H(s, 1) = b$ and
  $\forall s \in [0, 1] : t \mapsto H(s, t)$ is a *continuous curve* from a to b and
  when s *varies* from 0 to 1, $t \mapsto H(s, t)$ *varies* from γ_0 to γ_1.
  Intuitively G is an *"area without holes"*. ⟨p. 41 ∈ [1]⟩
- $G \subset \mathbb{C}$ **is *starshaped* around** $a \in G$ $\stackrel{def}{=}$
  $\forall z \in G$: the *line* from a to z is *contained* in G. ⟨p. 41 ∈ [1]⟩
- $G \subset \mathbb{C}$ **is *convex*** $\stackrel{def}{=}$
  $\forall a, b \in G$: the *line* from a to b is *contained* in G. ⟨p. 41 ∈ [1]⟩
  I.e.: $\forall t \in [0, 1] : \gamma(t) = (1 - t)a + tb \in G$.
- $A \subset \mathbb{C}$ ***curve connected*** $\stackrel{def}{=}$
  for *any* 2 points z_1, z_2 in A, z_1 and z_2 can be *connected* by a *continuous curve* $\gamma : [0, 1] \to \mathbb{C}$ in A.
  I.e.: $\gamma(0) = z_1, \gamma(1) = z_2, \forall t \in [0, 1] : \gamma(t) \in A$. ⟨def. 5.1 p. 73 ∈ [1]⟩

2.1 Comparison of Areas

- $G \subset \mathbb{C}$ *starshaped* $\Rightarrow$ G *enkeltsammenhængende.* ⟨p. 41 ∈ [1]⟩
- $G \subset \mathbb{C}$ *convex* $\Rightarrow$ G *starshaped* around *any of it's points*, and thus also *enkeltsammenhængende.* ⟨p. 41 ∈ [1]⟩
- $A \subset \mathbb{C}$ *star shaped* around $p_0 \in A \Rightarrow A$ *curve connected* ⟨p. 73 ∈ [1]⟩
- $A \underset{conxex}{\subset} \mathbb{C} \Rightarrow A$ *curve connected* ⟨p. 73 ∈ [1]⟩
- $A \underset{area}{\subset} \mathbb{C} \Rightarrow A$ *curve connected* ⟨p. 73 ∈ [1]⟩
- A *curve connected* and *open* $\Rightarrow$ ⟨p. 74 ∈ [1]⟩
  any 2 points in A can be *connected* by a *stairline* (i.e. A is an *area*)

2.2 Concrete Examples of Areas in $\mathbb{C}$ (or $\mathbb{R}^2$)

- $\mathbb{C}$ with a *halfline removed* $\Rightarrow$
  starshaped around *any point* of the *opposite halfline* ⟨p. 42 ∈ [1]⟩
  A.k.a. the *sliced plane* of *angle* $\alpha \in \mathbb{R}$: $\mathbb{C}_\alpha = \mathbb{C} \backslash \{re^{i\alpha} \mid r \geq 0\}$ ⟨p. 77 ∈ [1]⟩
  In particular: $\mathbb{C}_\pi = \mathbb{C} \backslash \{z \in \mathbb{R} \mid z \leq 0\}$
- An *angle space* is:
  starshaped, but *not convex* if the *angle* $> 180°$ ⟨p. 42 ∈ [1]⟩
- The *area outside* a *parabola* is:
  enkeltsammenhængende, but *not starshaped* ⟨p. 42 ∈ [1]⟩
- $\mathbb{C}$ with a *point removed* or one or more *closed discs* $\Rightarrow$
  not enkeltsammenhængende ⟨p. 42 ∈ [1]⟩
- The *circumference* of a *circle* is *curve connected* ⟨p. 74 ∈ [1]⟩

2.3 Misc Theorems About Areas

- For $G \underset{enkeltsammenhngende\ area}{\subset} \mathbb{C}$ and $D = \{a_1, \ldots, a_m\} \subset G$ a set of *points* in G and $E \subset D, E \neq \emptyset$: ⟨p. 125 ∈ [1]⟩
  There *exists* $G_1 \underset{enkeltsammenhngende\ subarea}{\subset} G$ such that $D \cap G_1 = E$.

3 Curves in $\mathbb{C}$ (or $\mathbb{R}^2$)

Premises: $\gamma : [a, b] \to \mathbb{C}$ ***parametrization* of *oriented continuous curve***

- **An *(oriented) continuous curve* in** $\mathbb{C}$ given by a parametrization $\stackrel{def}{=}$ a *map* $f : [a, b] \to \mathbb{C}$ of a *closed, limited interval* $[a, b]$ into $\mathbb{C}$. ⟨p. 30 ∈ [1]⟩
  $f(a)$ is the *starting point*, $f(b)$ is the *end point* and f is the *parametrization*.

- **An *(oriented) continuous curve* in** $\mathbb{C}$ $\stackrel{def}{=}$
  the *equivalence class* of *curve parametrizations*, where 2 curves $\gamma : [a, b] \to \mathbb{C}$, $\tau : [c, d] \to \mathbb{C}$ are *equivalent* if there *exists* a *strictly growing map* $\phi : [a, b] \to [c, d]$, such that $\tau \circ \phi = \gamma$ - i.e. the *equivalence class* w.r.t. *orientation preserving reparametrizations*. ⟨p. 30 ∈ [1]⟩

- **The *opposite curve* of an *(oriented) continuous curve*** $\gamma \stackrel{def}{=}$
  $t \mapsto \gamma(a + b - t)$ ⟨p. 30 ∈ [1]⟩

- **A *closed (oriented) continuous curve*** γ **in** $\mathbb{C}$ $\stackrel{def}{=}$
  $\gamma(a) = \gamma(b)$ ⟨p. 30 ∈ [1]⟩

- **The set of *curve points*** γ^* **on a *curve*** $\gamma \stackrel{def}{=} \gamma^* = \gamma([a, b])$ ⟨p. 30 ∈ [1]⟩

- **A *simple closed oriented continuous curve* / *Jordan curve*** $\gamma \stackrel{def}{=}$
  an *oriented closed continuous curve* $\gamma : [a, b] \to \mathbb{C}$,
  which does *not intersect itself*, i.e. $\gamma|_{[a,b[}$ is *injective* ⟨p. 31 ∈ [1]⟩
  We will *usually orient* such a curve *positively*, i.e. *counter clockwise*, so that the *inner area* of the curve is *to the left* when running through the curve.

- *Jordan's Curve Theorem*: A *Jordan curve* splits $\mathbb{C}$ into 2 *areas*:
  An *inner limited area* and an *outer unlimited area*, both of which have the *curve as their edge*. ⟨p. 31 ∈ [1]⟩

- ***Smooth curve* / C^1*-curve*** $\gamma \stackrel{def}{=} \gamma$ is *continuously differentiable*,
  i.e. *differentiable* and the *derivative* γ' is *continuous* ⟨p. 31 ∈ [1]⟩

- **The *tangent* of an *oriented continuous curve*** $\stackrel{def}{=} \gamma'(t)$ ⟨p. 31 ∈ [1]⟩

- **The *speed* of an *oriented continuous curve*** $\stackrel{def}{=} |\gamma'(t)|$ ⟨p. 31 ∈ [1]⟩

3.1 Concrete Examples of Curves

- The *circle* with *center* $(0, 0)$ and *one counter clockwise traversal* $\stackrel{def}{=}$
  $C_r = re^{it}$ for $t \in [0, 2\pi]$ ⟨ex. 2.14 p. 38 ∈ [1]⟩

- The *circle* with *center* a and *one counter clockwise traversal* $\stackrel{def}{=}$
  $\partial K(a, r)$

- A *line* at *angle* $\alpha \in \mathbb{R}$ through 0 $\stackrel{def}{=} \{se^{i\alpha} \mid s \in \mathbb{R}\}$ ⟨p. 78 ∈ [1]⟩

3.2 Composite Curves

Premises: $\gamma : [a, b] \to \mathbb{C}, \delta : [c, d] \to \mathbb{C}$ ***oriented C^1-curves***

- **The *composite curve* $\gamma \cup \delta$ of γ and δ**, where $\gamma(b) = \delta(c) \overset{def}{=}$
  $(\gamma \cup \delta)(t) = \begin{cases} \gamma(t), & t \in [a, b] \\ \delta(t + c - b), & t \in [b, b + (d - c)] \end{cases}$.
  $(\gamma \cup \delta) : [a, b + (d - c)] \to \mathbb{C}$ is *piecewise* C^1, since:
  $(\gamma \cup \delta)|_{[a,b]}$ is C^1 and $(\gamma \cup \delta)|_{[b,b+(d-c)]}$ is C^1.

- **γ is *piecewise C^1* / *piecewise smooth*** $\overset{def}{=}$
  there *exists separation points* $a = t_0 < t_1 < \ldots < t_n = b$ such that
  $\forall j \in \{1, \ldots, n\} : \gamma_j = \gamma|_{[t_{j-1},t_j]}$ are C^1 ⟨def. 2.5 p. 33 ∈ [1]⟩

- **A *contour* / *path* γ** $\overset{def}{=}$
  γ is *piecewise* C^1 ⟨def 2.5 p. 33 ∈ [1]⟩

3.3 Curve Integrals

Premises: $\gamma : [a, b] \to \mathbb{C}$ ***oriented C^1-curve***, $f : \gamma^* \to \mathbb{C}$ ***continuous***

- **The *curve integral* of f *along* γ** $\overset{def}{=}$
  $\int_\gamma f = \int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$ ⟨def 2.2 p. 31 ∈ [1]⟩

- $\int_\gamma f$ does *not change*, when γ is *reparametrized* into $\gamma \circ \phi$, where
  $\phi : [c, d] \to [a, b]$ is a *bijective C^1-map* and $\forall t : \phi'(t) > 0$ ⟨rem. 2.3 p. 31 ∈ [1]⟩

- If $-\gamma$ is the *opposite curve* of γ, then:
  $\int_{-\gamma} f = -\int_\gamma f$ ⟨rem. 2.3 p. 31 ∈ [1]⟩

- By writing $z = x+iy, f(z) = u(x, y)+iv(x, y), \gamma(t) = x(t)+iy(t)$ we have:
  $\int_\gamma f = \int_\gamma udx - vdy + i\int_\gamma vdx + udy = \int_\gamma (u, -v) \cdot ds + i\int_\gamma (v, u) \cdot ds$, where
  $u = u(x(t), y(t)), v = v(x(t), y(t)), dx = x'(t), dy = y'(t)$ ⟨rem. 2.3 p. 31-32 ∈ [1]⟩

3.4 Curve Integrals of Composite Curves and Contours / Paths

Premises: $\gamma : [a, b] \to \mathbb{C}, \delta : [c, d] \to \mathbb{C}$ ***oriented C^1-curves***
$f : \gamma^* \to \mathbb{C}$ ***continuous***

- For $g : (\gamma \cup \delta)^* \to \mathbb{C}$ *continuous*: $\int_{\gamma\cup\delta} g = \int_\gamma g + \int_\delta g$ ⟨p. 32 ∈ [1]⟩

- For *contour* γ: **The *curve integral* of f along the *contour* γ** $\overset{def}{=}$
  $\int_\gamma f = \int_{\gamma_1\cup\cdots\cup\gamma_n} f = \int_{j=1}^n \int_{\gamma_j} f = \sum_{j=1}^n \int_{\gamma_{j-1}}^{\gamma_j} f(\gamma(t))\gamma'(t)dt$
  ⟨def. 2.5 p. 33 ∈ [1]⟩

- For *contour* γ: $f(\gamma(t))\gamma'(t)$ is *piecewise continuous* for $t \in [a, b]$ and thus *Riemann Integrable*, so the *curve integral* long the *contour* γ can be seen as the *definition* of the *integral*:
  $\int_a^b f(\gamma(t))\gamma'(t)dt$. ⟨rem. 2.6 ∈ [1]⟩

- **The *length* $L(\gamma)$ of a *contour*** $\gamma(t) = x(t) + iy(t) \overset{def}{=}$
  $L(\gamma) = \int_a^b |\gamma'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$ ⟨def. 2.7 p. 34 ∈ [1]⟩

- *Estimation Lemma*: For a *contour* γ:
  $|\int_\gamma f| \leq \max_{z \in \gamma^*} |f(z)| L(\gamma)$ ⟨lemma 2.8 p. 34 ∈ [1]⟩

- If we can *find* K, such that $\forall z \in \gamma^* : |f(z)| \leq K$, then:
  $|\int_\gamma| \leq KL(\gamma)$ ⟨rem. 2.9 p. 34 ∈ [1]⟩

3.5 Rules for Curve Integrals and Uniform Convergence

(FIXME: move to Theorems from Metric Spaces and Topology?)

- Let $\gamma : [a, b] \to \mathbb{C}$ be a *contour* in $\mathbb{C}$ and let $f_n : \gamma^* \to \mathbb{C}$ be a *sequence* of *continuous functions* then:
  - If $f_n \to f$ *uniformly* on γ^* we have:
    $\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f = \int_\gamma \lim_{n\to\infty} f_n$ ⟨thm. 4.6 p. 59 ∈ [1]⟩
  - If $\sum_{n=0}^{\infty} f_n$ *converges uniformly* on γ^* with *sum function* $s : \gamma^* \to \mathbb{C}$:
    $\sum_{n=0}^{\infty} \int_\gamma f_n = \int_\gamma f = \int_\gamma \sum_{n=0}^{\infty} f_n$ ⟨thm. 4.6 p. 59 ∈ [1]⟩

3.6 Curve Theorems

- *The tile lemma*: $G \underset{open}{\subset} \mathbb{C}$, $\gamma : [a, b] \to \mathbb{C}$ *continuous curve* in G.
  There *exists* finitely many *splitting points* $a = t_0 < t_1, < \cdots < t_n = b$ in $[a, b]$ and a *radius* $r > 0$ such that
  $\cup_{i=0}^{n} K(\gamma(t_i), r) \subset G$ and such that
  $\forall i \in \{0, 1, \ldots, n\} : \gamma([t_i, t_{i+1}]) \subset K(\gamma(t_i), r)$ ⟨lemma 5.2 p. 74 ∈ [1]⟩

4 Holomorphic Functions

Definitions

Premises: $G \underset{open}{\subseteq} \mathbb{C}, f : G \to \mathbb{C}$

- f ***(complex) differentiable*** **in** $z_0 \in G \underset{def}{=} \frac{f(z_0+h)-f(z_0)}{h}$ has a limit for $h \to 0$.
  This is called $f'(z_0)$, the *differential quotient* of f in z_0 ⟨def. 1.1, p. 9 ∈ [1]⟩
- f ***holomorphic*** $\overset{def}{=} \forall z \in G : f$ is *(complex) differentialble* in z.
  Then $f' : G \to \mathbb{C}$ is the *differential quotient* or *derived function* ⟨def. 1.1, p. 9 ∈ [1]⟩
- $\mathcal{H}(G) \underset{def}{=}$ the *set of holomorphic functions* $f : G \to \mathbb{C}$ ⟨def. 1.1, p. 9 ∈ [1]⟩
- f ***conformal*** **in** $z_0 \underset{def}{=} f$ *preserves angles* between *curves* passing through z_0 ⟨p. 13 ∈ [1]⟩
- f ***biholomorphic*** $\overset{def}{=} f$ *holomorphic* and f^{-1} *holomorphic* ⟨p. 86 ∈ [1]⟩

Theorems

Premises: $G \underset{open}{\subseteq} \mathbb{C}, f, g : G \to \mathbb{C}, z_0 \in G, a \in \mathbb{C}$

- f *differentialble* in z_0 with $f'(z_0) = a \Leftrightarrow f(z_0 + h) = f(z_0) + ha + h\epsilon(h)$ for $h \in K'(0, r) \subseteq G$ (rem. 1.2, p. 9 [1])
- f *differentialble* in $z_0 \Rightarrow f$ *continuous* in z_0 (p. 10 [1])
- For f, g *(complex) differentiable* in z_0:
  $$\begin{aligned} (af)'(z_0) &= af'(z_0) \\ (f \pm g)'(z_0) &= f'(z_0) \pm g'(z_0) \\ (fg)'(z_0) &= f(z_0)g'(z_0) + f'(z_0)g(z_0) \\ (\tfrac{f}{g})'(z_0) &= \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}, \text{ for } g(z_0) \neq 0 \end{aligned} \quad \text{(p. 10 [1])}$$
- $\mathcal{H}(G)$ is *stable* under *addition, subtraction, multiplication* and *division* if the denominator is never 0.
  I.e. $\mathcal{H}(G)$ is a *complex vector space* and a *commutative ring* (thm. 1.3, p. 10 [1])
- Differentiation rules for some *holomorphic functions*:
  $$\begin{aligned} &\tfrac{d}{dz}(z^n) = nz^{n-1} \\ &\tfrac{d}{dz}(z^{-n}) = -nz^{-n-1} \quad \text{holomorphic in } \mathbb{C}\backslash\{0\} \end{aligned} \quad \text{(p. 10 [1])}$$
- *Function composition* for $f : G \to \mathbb{C}$ differentiable in $g(z_0) \in G, g : U \to \mathbb{C}$ differentiable in $z_0 \in U$ and $g(U) \subseteq G$. U can be an open interval (g is a usual differentiable function) or an open set in $\mathbb{C}$ (g is complex differentiable in U):
  $(f \circ g)'(z_0) = f'(g(z_0))g'(z_0)$ (p. 11 [1])

- f *injective* and *holomorphic* in $G \Rightarrow$
  $f(G)$ *open* in $\mathbb{C}$
  $f^{\circ -1} : f(G) \to G$ *holomorphic*
  $f'(z) \neq 0$ for all $z \in G$ (thm. 1.4, p. 12 [1])
  $(f^{\circ -1})' = \frac{1}{f' \circ f^{\circ -1}}$ for all $z \in G$
  $(f^{\circ -1})'(f(z)) = \frac{1}{f'(z)}$ for all $z \in G$

- $f = u + iv$ *complex differentiable* in $z_0 = x_0 + iy_0 \Leftrightarrow u$ and v *differentiable* in (x_0, y_0) and the following (*Cauchy-Riemanns differential equations*) hold:
  $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$ and
  $\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$
  And then $f'(z_0) = \frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0)$ (thm. 1.6, p. 14 [1])

- The *Jacobian* J of $f = (x, y) \mapsto (u, v)$ is:
  $J = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$. For f *complex differentiable* the *Jacobian determinant* $\det J$ is: $\det J = \left[\frac{\partial u}{\partial x} \right]^2 + \left[\frac{\partial v}{\partial x} \right]^2 = \left[\frac{\partial u}{\partial y} \right]^2 + \left[\frac{\partial v}{\partial y} \right]^2 = |f'|^2$
  (rem. 1.7 p. 15-16 [1])

- The following are *holomorphic* in $\mathbb{C}$:
  $\exp(x)$ (p. 19 [1])
  $\cos(x)$
  $\sin(x)$
  $\tan(x), x \in \mathbb{C} \setminus \{\frac{\pi}{2} + \pi\mathbb{Z}\}$ (p. 23 [1])
  $\cot(x), x \in \mathbb{C} \setminus \pi\mathbb{Z}$ (p. 23 [1])
  $\cosh(x), x \in \mathbb{C}$ (p. 24 [1])
  $\sinh(x), x \in \mathbb{C}$ (p. 24 [1])
  $\tanh(x), x \in \mathbb{C} \setminus i\{\frac{\pi}{2} + \pi\mathbb{Z}\}$ (p. 24 [1])
  $\coth(x), x \in \mathbb{C} \setminus i\pi\mathbb{Z}$ (p. 24 [1])

Premises: $G \underset{open}{\subseteq}_{\textbf{area}} \mathbb{C}, f : G \to \mathbb{C}$ ***holomorphic***

- $\forall z \in G : f'(z) = 0 \Rightarrow f$ *constant* (thm. 1.9, p. 16 [1])
- f has only *real values* $\Rightarrow f$ *constant* (cor. 1.10 p. 16 [1])

4.1 Geometric Meaning when $f'(z_0) \neq 0$

Premises: $G \underset{open}{\subseteq} \mathbb{C}, f : G \to \mathbb{C}$ **holomorphic,** $z_0 \in G, f'(z_0) \neq 0$**.**
Notation: $w = f(z), w_0 = f(z_0), f'(z_0) = r(\cos\theta + i\sin\theta)$**.**

- For *differentiable curve* $\gamma : I \to G$ through $z_0 = \gamma(t_0)$ where $\gamma'(t_0) \neq 0$:
  $f(\gamma(t))$ differentiable curve through w_0
  $\gamma(t)$ has a tangent in z_0 parallel to $\gamma'(t_0)$ (considered as the vector $(Re(\gamma'(t_0)), Im(\gamma'(t_0)))$)
  Parametrized tangent for $\gamma(t)$: $z_0 + s\gamma'(t_0), s \in \mathbb{R}$
  Parametrized tangent for $f(\gamma(t))$: $w_0 + sf'(z_0)\gamma'(t_0), s \in \mathbb{R}$
  The speed vector of $f(\gamma(t_0))$ is $\gamma'(t_0)$ *streched* by r and *rotated* by θ
  For some other curve $\widetilde{\gamma}(t)$: $\angle\gamma(t_0), \widetilde{\gamma}(t_0) = \alpha \Rightarrow \angle f(\gamma(t_0)), f(\widetilde{\gamma}(t_0)) = \alpha$ (f *conformal* in z_0)
  Points close to and on the *left* of $\gamma(t)$ *remains to the left* of $f(\gamma(t))$ through f

(p. 12-14 [1])

- f *conformal* in z_0 (p. 13 [1])
- f has *stretching factor* $r = |f'(z_0)|$ and *rotation angle* θ in z_0 (p. 13 [1])

5 Integrability, Integration, Differentiation etc.

Premises: $G \underset{area}{\subset} \mathbb{C}, f : G \to \mathbb{C}$

- $F : G \to \mathbb{C}$ **is a ⟨ DK: ⟩ *stamfunktion* of** $f : G \to \mathbb{C} \overset{def}{=}$ F is *holomorphic* in G and $F' = f$. ⟨def. 2.10 p. 35 ∈ [1]⟩
  Furthermore, $\forall k \in C : F + k$ is also a *stamfunktion* of f.

- For f *continuous*, the *following are equivalent*:
  - f has a *stamfunktion* $F(x) = \int_\gamma f$ ⟨thm. 2.13 p. 38 ∈ [1]⟩
  - $\int_\gamma f = 0$ for *any closed contour* γ in G ⟨thm. 2.13 p. 38 ∈ [1]⟩
  - For *any* $z_1, z_2 \in G$: ⟨thm. 2.13 p. 38 ∈ [1]⟩
    $\int_\gamma f$ has the *same value* for *any contour* from z_1 to z_2

- If a *continuous function* $f : G \to \mathbb{C}$ has a *stamfunktion* $F : G \to \mathbb{C}$ then:
  $\int_\gamma f(x)dx = F(x_2) - F(x_1)$ for *any contour* from x_1 to x_2. ⟨thm. 2.11 p. 35 ∈ [1]⟩

- For $z_0 \in G$ and $f \in \mathbb{H}(G \backslash \{z_0\})$ and a *simple closed contour* C in G, which *sorrounds* z_0 *counter clockwise*, we can often find *another simple closed contour* K in G which *sorrounds* z_0 *counter clockwise*, such that:
  $\int_C f(z)dz = \int_K f(z)dz$ ⟨p. 46 ∈ [1]⟩
  Method: Find *finite number of* "snit" of *star shaped areas* between C and K. The theorem below is an example. ⟨p. 47 ∈ [1]⟩

- For $f \in \mathcal{H}(G \backslash \{z_0\})$ and *curves* $C = \partial K(a, r)$ and $K = \partial K(z_0, s)$, where $\overline{K(z_0, s)} \subset K(a, r)$, $\overline{K(a, r)} \subset K(a, R) \subset G$, we have:
  $\int_{\partial K(a,r)} f = \int_{\partial K(z_0,s)} f$ ⟨ex. 3.7 p. 47-48 ∈ [1]⟩

5.1 Integrability, Integration, Differentiation etc. for Holomorphic Functions

Premises: $G \subset \mathbb{C}, f : G \to \mathbb{C}$ ***holomorphic*** **(i.e.** $f \in \mathcal{H}(G)$**)**

- *Cauchy's Integral Theorem*: For G *enkeltsammenhængende area* and any *closed contour* γ in G we have:
  $\int_\gamma f(z)dz = 0$ ⟨thm. 3.1 p. 42, thm. 3.3 p. 45 ∈ [1]⟩

- *Cauchy's Integral Formula*: For $G \underset{open}{\subset} \mathbb{C}$ and $\overline{K(a, r)} \subset G$, then:
  $\forall z_0 \in K(a, r) : f(z_0) = \frac{1}{2\pi i} \int_{\partial K(a,r)} \frac{f(z)}{z - z_0} dz$
  Intuition: If we *know* the *value* of f *on a circle*, the the *value* of *all points inside that circle* are *known too.* ⟨thm. 3.8 p. 48 ∈ [1]⟩

- Let $G \underset{open}{\subset} \mathbb{C}$ and $\overline{K(a, r)} \subset G$, then:
  $f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})d\theta$ ⟨cor. 3.9 p. 49 ∈ [1]⟩
  Intuition: The *value* at the *center of a circle* is the *mean value* of the *values at the circumference*

- *Goursat's Lemma*: For $G \underset{open}{\subset} \mathbb{C}$:
  $\int_{\partial\triangle} f(z)dz = 0$ for *any (massive) triangle* $\triangle \subset G$. ⟨lemma 3.2 p. 43 ∈ [1]⟩

- $G \underset{open}{\subset} \mathbb{C} \Rightarrow f'$ *continuous* ⟨p. 42-43 ∈ [1]⟩

- $G \underset{enkeltsammenhngende\ area}{\subset} \mathbb{C} \Rightarrow f$ is *integrable* ⟨thm. 3.5 p. 46 ∈ [1]⟩

- f has a *stamfunktion* in $G \Rightarrow$
  for *any closed contour* γ in G: $\int_\gamma f(z)dz = 0$ ⟨rem. 3.4 p. 46 ∈ [1]⟩

- If $G_1 \underset{enkeltsammenhngende\ area}{\subset} G$, them:
  for *any closed contour* γ *contained in* G_1: $\int_\gamma f(z)dz = 0$ ⟨rem. 3.4 p. 46 ∈ [1]⟩

5.2 Concrete Examples of Integrability

- $\int_{\partial K(a,r)} \frac{1}{z}dz = 0$, for $a \neq 0, r < |a|$. So $\frac{1}{z}$ is *holomorphic* in a *non-enkeltsammenhængende area* which is *contained* in a *half-plane* (i.e. an *enkeltsammenhængende subarea*) ⟨rem. 3.4 p. 46 ∈ [1]⟩

5.3 Concrete Holomorphic Functions and Their Derivatives and Stamfunktioner

Name	Properties
Power Function	$f(x) = x^n : \mathbb{C} \to \mathbb{C}$ for $n \in \mathbb{Z}$
⟨ex. 2.14 p. 38 ∈ [1]⟩	$\int_{C_r} f(x)dx = \begin{cases} 0, & n \neq (-1) \\ 2\pi i, & n = (-1) \end{cases}$
	So x^{-1} does *not* have a *stamfunction*!
Polynomium	$p(x) : \mathbb{C} \to \mathbb{C} = a_0 + a_1 x + \cdots + a_n x^n = \sum_{k=0}^n a_k x^k$, for $a_k \in \mathbb{C}$
⟨p. 11, 35 ∈ [1]⟩	$P(x) = k + a_0 x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}$
	$p'(z) = \sum_{k=1}^n k a_k z^{k-1}$

5.4 Power Series

Premises: $\sum_{n=0}^\infty a_n z^n$ **converges with convergence radius** $\rho > 0 (\rho \in]0, \infty]), f(z) = \sum_{n=0}^\infty a_n z^n$ **for** $|z| < \rho, K(0, \rho)$ **is** f**'s circle of convergence**

- **Radius of Convergence** ρ **for** $\sum_0^\infty a_n z^n \overset{def}{=}$
  $\rho = \sup T$, where $T = \{t \geq 0 \mid \{|a_n| t^n\}$ is *limited*$\}$ ⟨p. 17 ∈ [1]⟩

- f and f's *component-wise derived* series has the *same radius of convergence* ⟨lemma 1.11, p. 17 ∈ [1]⟩

- f is *holomorphic* inside $K(0, \rho)$ ⟨thm. 1.12, p. 17 ∈ [1]⟩

- f can be *complex derived any number of times* in $K(0, \rho)$ and:
  $a_k = \frac{f^{(k)}(0)}{k!}, k \in \{0, 1, ...\}$. So f is it's own *Taylor Series* around 0:
  $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} z^n, |z| < \rho$ ⟨cor. 1.13, p. 19 ∈ [1]⟩

- *Stamfunktion*: $F(z) = k + \sum_{n=0}^\infty \frac{a_n}{n+1} z^{n+1}$ with *same radius of convergence* ρ. ⟨def. 2.10 p. 35 ∈ [1]⟩

- If $g(z) = \sum_0^\infty b_n z^n$ has *radius of convergence* $\rho' > 0$ and there *exists* $\rho'' \leq \min(\rho, \rho')$ such that $f(z) = g(z)$ for $|z| < \rho''$, then:
  $a_n = b_n$ for all n. (*Identity Thm.* for Power Series, ⟨thm. 1.14, p. 19 ∈ [1]⟩)

- $\sum_{n=0}^{\infty} a_n z^n$ *converges uniformly* towards f on *any closed disc* $\overline{K(0,r)}$ where $r < \rho$. ⟨thm. 4.5 p. 58 ∈ [1]⟩
  Not necessarily *uniform* convergence on $K(0,\rho)$ though!

5.5 Development of Holomorphic Functions in Power Series

- For any $G \underset{open}{\subset} \mathbb{C}$ **the *greatest open disc*** $K(a,\rho)$ **in** $G \overset{def}{=}$
  For $G = \mathbb{C}$: $K(a,\infty) = \mathbb{C}$
  For $G \neq \mathbb{C}$: $\rho = \inf\{|a-z| \mid z \in \mathbb{C}\backslash G\}$ ⟨p. 60 ∈ [1]⟩

- For $G \underset{open}{\subset} \mathbb{C}$: Any $f \in \mathcal{H}(G)$ is *arbitrarily often differentiable* and the *Taylor Series* with *center* $a \in G$ *converges* towards f in the *greatest open disc* $K(a,\rho) \subset G$:
  $\forall z \in K(a,\rho) : f(z) = \sum_{0=n}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$
  For $\overline{K(a,r)} \subset G$ and $z_0 \in K(a,r)$, *Cauchy's Integral Formula* holds for the n'th *derivative* for $n \in \mathbb{N}_0$:
  $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial K(a,r)} \frac{f(z)}{(z-z_0)^{n+1}}dz$ ⟨thm. 4.8 p. 60 ∈ [1]⟩

- For $G \underset{area}{\subset} \mathbb{C}$, $f : G \to \mathbb{C}$ we have:
  - f **is *(complex) analytical*** $\overset{def}{=}$
    $\forall a \in G$: there *exists* a *power series* $\sum_{n=0}^{\infty} a_n(z-a)^n$ with *sum* $f(z)$ in a *disc* $K(a,r) \subset G$ ⟨rem. 4.9 p. 62 ∈ [1]⟩
  - f *(complex) analytical* $\Leftrightarrow$ f *holomorphic* ⟨rem. 4.9 p. 62 ∈ [1]⟩
  - f *holomorphic* $\Rightarrow$ f' *continuous* and *holomorphic* ⟨rem. 4.9 p. 62 ∈ [1]⟩

5.6 Harmonic Functions

- For $u,v : G \to \mathbb{R}$ *differentiable* and *Cauchy-Riemann's equations* hold $\Rightarrow$ u,v are C^∞ in $G \underset{open}{\subset} \mathbb{R}^2$ ⟨p. 62 ∈ [1]⟩

- **The *Laplacian Operator*** $\triangle \overset{def}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
  I.e.: $\triangle\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$ ⟨p. 63 ∈ [1]⟩

- For $G \underset{open}{\subset} \mathbb{C}$. $\phi : G \to \mathbb{R}$ ***harmonic* in** $G \overset{def}{=} \triangle\phi = 0$ in G ⟨p. 63 ∈ [1]⟩

- For *holomorphic function* $f \Rightarrow$ Re f and Im f *harmohic* ⟨p. 63 ∈ [1]⟩

- For $G \underset{enkeltsammenhngende\ area}{\subset} \mathbb{R}^2 \approx \mathbb{C}$ and $u : G \to \mathbb{R}$ a C^2-function: there *exists* a *holomorphic* function $f : G \to \mathbb{C}$ with Re $f = u$ and f is *predetermined except* for a *purely imaginary constant* k ⟨thm. 4.10 p. 63 ∈ [1]⟩
  $v =$ Im f is a *conjugated harmonic function* to u

- For $G \underset{open}{\subset} \mathbb{C}$, $u \in C^2(G)$ and $\triangle u = 0 \Rightarrow u \in C^\infty(G)$ ⟨p. 63 ∈ [1]⟩

5.7 Morera's Theorem and Local Uniform Convergence

- For $G \underset{area}{\subset} \mathbb{C}, f : G \to \mathbb{C}$:
  f has *stamfunktion* $\Rightarrow$ f *holomorphic* ⟨thm. 4.11 p. 64 ∈ [1]⟩
  Remember: f does *not* necessarily have *stamfunktion* in *area*,
  but it *has* in *enkeltsammenhængende area*!

- *Morera's Theorem*: For $G \underset{area}{\subset} \mathbb{C}, f : G \to \mathbb{C}$ *continuous.*
  If $\int_\gamma f = 0$ for *any closed contour* γ or just
  $\int_{\partial\triangle} f = 0$ for *any triangle* $\triangle \subset G$, then
  f *holomorphic* in G ⟨thm. 4.12 p. 64 ∈ [1]⟩

- For $G \underset{open}{\subset} \mathbb{C}$. **A *sequence* of *functions*** $f_n : G \to \mathbb{C}$ ***converges locally uniformly* towards** $f : G \to \mathbb{C} \overset{def}{=}$
  $\forall a \in G : \exists r > 0 : \overline{K(a,r)} \subset G$ and such that
  $f_n(z) \to f(z)$ *uniformly* for $z \in \overline{K(a,r)}$ ⟨def. 4.14 p. 64 ∈ [1]⟩

- $G \underset{open}{\subset} \mathbb{C}, f_n : G \to \mathbb{C}$ *converges locally uniformly* towards $f \Rightarrow$
  f_n converges *pointwise* towards f ⟨p. 65 ∈ [1]⟩

- $G \underset{open}{\subset} \mathbb{C}$. **A *compact uniform sequence* of *functions*** $f_n : G \to \mathbb{C} \overset{def}{=}$
  $\forall K \underset{closed,limited}{\subset} G : f_n(z) \to f(z)$ *uniformly* for $z \in K$ ⟨rem. 4.16 p. 65 ∈ [1]⟩

- $F \underset{open}{\subset} \mathbb{C}$. A *sequence of functions* $f_n : G \to \mathbb{C}$ *converges locally uniformly* towards $f : G \to \mathbb{C} \Leftrightarrow$
  f_n *converges compact uniformly* towards f ⟨thm. 4.15 p. 65 ∈ [1]⟩

- $G \underset{open}{\subset} \mathbb{C}$. If a *sequence* $f_1, f_2, \dots$ from $\mathcal{H}(G)$ *converges locally uniformly* on G towards f, then $f \in \mathcal{H}(G)$
  Also the *sequence* $f_1', f_2', \dots$ *converges locally uniformly* on G towards f'.
  In general for the k'th *derivative* it holds:
  $\forall k \in \mathbb{N} : f_n^{(k)} \to f^{(k)}$ *locally uniformly* on G ⟨thm. 4.17, rem. 4.18 p. 65 ∈ [1]⟩
  ⟨thm. 1.12 is a special case of this ∈ [1]⟩

5.8 Entire Functions and Liouville's Theorem

- f **is *entire*** $\stackrel{def}{=} f : \mathbb{C} \to \mathbb{C}$ *holomorphic.*
  E.g. *polynomials*, exp, sin, cos, sinh, cosh but *not* $\frac{1}{z}$.
  The *set* of *entire functions*: $\mathcal{H}(\mathbb{C})$ ("infinite polynomials").
  Entire functions are *represented* for *any* $z \in \mathbb{C}$ by it's *Taylor series* with an *arbitrary center* (e.g. 0), so it is *equivalent* to *power series* with center 0 and *radius of convergence* $\rho = \infty$. ⟨p. 66 ∈ [1]⟩

- *Picard's Theorem*: For a *non-constant entire function* $f : \mathbb{C} \to \mathbb{C}$, either $f(\mathbb{C}) = \mathbb{C}$ or $f(\mathbb{C}) = \mathbb{C} \setminus \{a\}$ for a *suitable* $a \in \mathbb{C}$.
  If f is *not* a *polynomium*, then $f^{-1}(\{w\})$ is an *infinite set* for all $w \in \mathbb{C}$ *except* for *at most one.* ⟨thm. 4.19 p. 66-67 ∈ [1]⟩

- *Liouville's Theorem*: A *limited entire function* is *constant* ⟨thm. 4.20 p.67 ∈ [1]⟩

- *Fundamental Theorem of Algebra*: Any *polynomium* $p(z) = \sum_{k=0}^{n} a_k z^k$ of *degree* $n \geq 1$ has *at least* one *root* in $\mathbb{C}$. ⟨thm. 4.21 p. 67 ∈ [1]⟩

- $\mathbb{C}[z] \stackrel{def}{=}$ the *set of polynomials* of *one complex variable* and *complex coefficients*: $p(z) = a_0 + a_1 z + \cdots + a_n z^n : \mathbb{C} \to \mathbb{C}$ ⟨p. 68 ∈ [1]⟩

- $\mathbb{C}[z] \subset \mathcal{H}(\mathbb{C})$ ⟨p. 68 ∈ [1]⟩

- For $d \in \mathbb{C}[z]$ *not* the *zero-polynomium.*
  For *any* $p \in \mathbb{C}[z]$ there *exists unique polynomials* $q, r \in \mathbb{C}[z]$ such that:
  $p = qd + r$, degree$(r) <$ degree(d).
  We give the 0-*polynomium* the *degree* ∞.
  d, q, r are called *divisor*, *quotient* and *rest* respectively.
  If d, p have *real coefficients*, then so does q and r. ⟨thm. 4.22 p. 68 ∈ [1]⟩

- A *polynomium* $p(z)$ of degree $n \geq 1$ has *exactly* n *roots* in $\mathbb{C}$ (counting multiplicity). ⟨thm. 4.22 p. 68 ∈ [1]⟩

- **An *entire trancendental function*** $\stackrel{def}{=}$
  an *entire function* which is *not* a *polynomial*,
  i.e. a *series* with *infinitely many non-zero coefficients.* ⟨ex. 6.22 p. 113 ∈ [1]⟩

6 Argument and Number of Revolutions

- **The *set of arguments* of $z \in \mathbb{C}$, $\arg z$** $\overset{def}{=}$ all $\theta \in \mathbb{R}$ such that $z = |z|e^{i\theta}$ ⟨p. 77 ∈ [1]⟩
- **The *main argument* of $z \in \mathbb{C}$, Arg z** $\overset{def}{=}$ the *unique argument* in $]-\pi, \pi]$ ⟨p. 77 ∈ [1]⟩
  Arg is *continuous* and C^∞ on the *sliced plane* C_π
- We have: Arg $z \in \arg z \cap \,]-\pi, \pi]$ and $\arg z =$ Arg $z + 2\pi\mathbb{Z}$ ⟨p. 77 ∈ [1]⟩
- Arccos: $[-1, 1] \to [0, \pi]$ is the *inverse* of $\cos : [0, \pi] \to [-1, 1]$ ⟨p. 77 ∈ [1]⟩
- Arctan: $\mathbb{R} \to \,]-\frac{\pi}{2}, \frac{\pi}{2}[$ is the *inverse* of $\tan : \,]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}$ ⟨p. 77 ∈ [1]⟩
- Arg $z =$ Arccos $\frac{x}{r}$ for $z = x + iy$, $y > 0$ and $r = |z|$ ⟨p. 77 ∈ [1]⟩
- Arg $z = -$ Arccos $\frac{x}{r}$ for $z = x + iy$, $y < 0$ and $r = |z|$ ⟨p. 77 ∈ [1]⟩
- Arg $z =$ Arctan $\frac{y}{x}$ for $z = x + iy$, $x > 0$ ⟨p. 77 ∈ [1]⟩
- **An *argument function* θ for a subset $A \subset \mathbb{C}\backslash\{0\}$** $\overset{def}{=}$ $\theta : A \to \mathbb{R}$ such that $\theta(z) \in \arg z$ for $z \in A$ or equivalently: $z = |z|e^{i\theta(z)}$ for $z \in A$ ⟨p. 77 ∈ [1]⟩
- For *any* $\alpha \in \mathbb{R}$: **The *argument function* Arg$_\alpha$** $\overset{def}{=}$ defined on the *sliced plane* $\mathbb{C}_\alpha$ by: $\text{Arg}_\alpha z \in \arg z \cap \,]\alpha - 2\pi, \alpha[$.
  $\text{Arg}_\alpha z$ is *continuous* and C^∞ on $\mathbb{C}_\alpha$.
  Also Arg $z = \text{Arg}_\pi z$ on $\mathbb{C}_\pi$ ⟨p. 77 ∈ [1]⟩
- For *any* $A \underset{curve\ connected}{\subset} \mathbb{C}\backslash\{0\}$:
  Let $\theta : A \to \mathbb{R}$ be a *continuous argument function.*
  Then for *any* $p \in \mathbb{Z}$, $\theta + 2\pi p$ is *also* a *continuous argument function* for A and there *are no others!* ⟨lemma 5.8 p. 78 ∈ [1]⟩
- Let $\gamma : [a, b] \to \mathbb{C}\backslash\{0\}$ be a *continuous curve*, which *doesn't* pass through 0, then there *exists* a *continuous argument function* $\theta : [a, b] \to \mathbb{R}$ along γ, and *any continuous argument function* along γ is given by $\theta(t) + 2\pi p$ for suitable (fixed) $p \in \mathbb{Z}$ ⟨lemma 5.9, thm. 5.10 p. 78-79 ∈ [1]⟩
  So we have $\forall t \in [a.b] : \theta(t) \in \arg \gamma(t)$ or equivalently:
  $\forall t \in [a, b] : \gamma(t) = |\gamma(t)|e^{i\theta(t)}$
  γ^* does *not necessarily* have a *continuous argument function* though!
- Let $\gamma : [a, b] \to \mathbb{C}\backslash\{0\}$ be a *continuous curve.*
  The *argument variation along* γ $\overset{def}{=}$
  argvar $(\gamma) = \theta(b) - \theta(a)$ for an *arbitrary argument function* θ along γ ⟨def. 5.12 p. 80 ∈ [1]⟩
- Let $\gamma : [a, b] \to \mathbb{C}\backslash\{0\}$ be a *closed continuous curve.*
  The *number of revolutions* around 0 $\overset{def}{=}$
  $\omega(\gamma, 0) = \frac{1}{2\pi} argvar(\gamma)$. $\omega(\gamma, 0) \in \mathbb{Z}$ ⟨def. 5.12 p. 80 ∈ [1]⟩

6.1 More About The Number of Revolutions

- For $\gamma : [a, b] \to \mathbb{C}\backslash\{0\}$ a *contour not passing* 0:
  $\int_\gamma \frac{1}{z} dz = \log|\frac{\gamma(b)}{\gamma(a)}| + i \ argvar(\gamma)$ ⟨thm. 5.23 p. 88 ∈ [1]⟩

- For $\gamma : [a, b] \to \mathbb{C}\backslash\{0\}$ a *closed contour not passing* 0:
  the *number of revolutions* around 0:
  $\omega(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} dz$ ⟨thm. 5.23 p. 88 ∈ [1]⟩

- The *number of revolution* around 0 is given as:
  $\omega(\gamma, 0) = \sum_{k=0}^{n-1} sign(A_k)$, where γ *intersects (non-parallelly) some halfline* from 0 at *points* $A_1, \dots, A_{n-1}$ and
  $sign(A_k) = 1$ when A_k is *passed counter-clockwise* and
  $sign(A_k) = -1$ when A_k is *passed clockwise* ⟨thm. 5.24 p. 89 ∈ [1]⟩

- **The *number of revolutions* of γ around z, $\omega(\gamma, z)$** $\overset{def}{=}$
  $\omega(\gamma, z) = \omega(\gamma - z, 0)$, where
  $\gamma - z = t \mapsto \gamma(t) - z$ and $\gamma - z : [a, b] \to \mathbb{C}\backslash\{0\}$ *does not pass* 0 ⟨p. 91 ∈ [1]⟩

- For $\gamma : [a, b] \to \mathbb{C}$ *closed contour*:
  $\omega(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} dz$ for any $z_0 \in \mathbb{C}\backslash\gamma^*$ ⟨p. 92 ∈ [1]⟩

- For $\gamma : [a, b] \to \mathbb{C}$ *closed continuous curve*:
  The *number of revolutions* $\omega(\gamma, \cdot) : \mathbb{C}\backslash\gamma^* \to \mathbb{Z}$
  is *constant* in *each of the components* of $\mathbb{C}\backslash\gamma^*$ ⟨thm. 5.25 p. 92 ∈ [1]⟩

7 N'th Roots

- Intuition: Think of z^n and $\sqrt[n]{z}$ as *angle scaling* functions.
  z^n *scales up* by factor n.
  $\sqrt[n]{z}$ *scales down* by n and *circularly repeats* itself n times.

- **The n'th roots of $z = re^{i\theta} \in \mathbb{C}$ where $r = |z|$, $\theta \in \mathbb{R}$** $\stackrel{def}{=}$
  $\sqrt[n]{0} = 0$ and $\sqrt[n]{z} = \{z_k \mid k \in \{0, 1, \ldots, n-1\}\}$ where
  $z_k = \sqrt[n]{r}e^{i\theta_k}$, $\theta_k = \frac{\theta}{n} + k\frac{2\pi}{n}$ for $k \in \{0, 1, \ldots, n-1\}$. ⟨p. 80-81 ∈ [1]⟩
  Notice: $\sqrt[n]{z} : \mathbb{C} \to \mathcal{P}(\mathbb{C})$ is a *multifunction.*

- A *branch*, f, of $\sqrt[n]{z}$ for a *subset* $A \subset \mathbb{C}$:
  $f : A \to \mathbb{C}$ such that $f(z) \in \sqrt[n]{z}$ for all $z \in A$ ⟨p. 81 ∈ [1]⟩

- For $A \subset \mathbb{C}$: If $f : A \to \mathbb{C}$ is a *branch of* $\sqrt[n]{z}$, then ⟨p. 81 ∈ [1]⟩
  $z \mapsto f(z)\exp(\frac{2\pi ik}{n})$ is *also* a *branch* of $\sqrt[n]{z}$ for $k \in \{0, 1, \ldots, n-1\}$

- 0 is a *branch point* of $\sqrt[n]{z}$ ⟨p. 81 ∈ [1]⟩

- For $n \in \mathbb{N}$: z^n maps the *angle space* $\{z \in \mathbb{C}\backslash\{0\} \mid |\text{Arg}z| < \frac{\pi}{n}\}$ *bijectively* onto $\mathbb{C}_\pi$.
  The *inverse function* $\rho_n(z) = \sqrt[n]{|z|}e^{i\frac{Arg\ z}{n}}$ is a *holomorphic branch* of $\sqrt[n]{z}$ on $\mathbb{C}_\pi$.
  $\rho_n'(z) = \frac{1}{n}\rho_n(z)^{1-n}$ ⟨thm. 5.13 p. 81 ∈ [1]⟩

- Description of *Riemann Surfaces* p. 81-82 ⟨p. 81-82 ∈ [1]⟩
  Multivalue functions can in general be *realized* on a *Riemann surface.*

8 Logarithm Function

- Intuition: The exp function *radialy* maps a *strip* of $\mathbb{C}$ into the *sliced* $\mathbb{C}_\alpha$.
- For $z \in \mathbb{C}$ and $z = u + iv$, the *solutions* $w \in \mathbb{C}$ to $\exp w = z$ are:
  $\log z = log|z| + i \arg z = \{log|z| + iv \mid v \in \arg z\}$ ⟨p. 82 ∈ [1]⟩
- The *value* of $\log z$ *corresponding* to the *main argument* of z is
  the *main logarithm* Log z. $\text{Log } z = \log|z| + i \text{ Arg } z$ ⟨p. 82 ∈ [1]⟩
- **The *inverse* of Log** z is $\exp|_{\{z \in \mathbb{C} | -\pi < Im\ \ z \leq \pi\}}$ ⟨p. 82 ∈ [1]⟩
- We have for $z > 0$: $\log z = \text{Log } z + 2\pi i\mathbb{Z}$ ⟨p. 82 ∈ [1]⟩
- **A *logarithm function* l for a *subset*** $A \subset \mathbb{C}\backslash\{0\} \overset{def}{=}$
  $l : A \to \mathbb{C}$ such that $l(z) \in \log z$ for $z \in A$ or equivalently:
  $\exp(l(z)) = z$ for $z \in A$ ⟨p. 82 ∈ [1]⟩
- If α is an *argument function* / *branch of the argument function* for $A \subset \mathbb{C}\backslash\{0\}$, then:
  $l(z) = \log|z| + i\alpha(z)$ is a *logarithm function* / a *branch of the logarithm function* for A,
  and if l is a *logarithm function* for A, then $\alpha = \text{Im } l$ is an *argument function* for A. ⟨p. 82-83 ∈ [1]⟩
  I.e.: A has a *logarithm function* ⇔ A has an *argument function*
- For $\alpha \in \mathbb{R}$, $\exp : \{z \in \mathbb{C} \mid \alpha - 2\pi < Im\ z < \alpha\} \to \mathbb{C}_\alpha$ is a *bijective continuous map* and the *inverse map* is:
  $\text{Log}_\alpha z = \log|z| + i \text{ Arg}_\alpha(z)$, which is *continuous* and *holomorphic* in $\mathbb{C}_\alpha$.
  $\frac{d}{dz} \text{ Log}_\alpha z = \frac{1}{\exp(Log_\alpha z)} = \frac{1}{z}$ ⟨p. 83 ∈ [1]⟩
- Log_α is a *holomorphic branch* of the *logarithm* in the *sliced plane* $\mathbb{C}_\alpha$ and *there* it is a *stamfunktion* to $\frac{1}{z}$ ⟨p. 83 ∈ [1]⟩
- Notice: $Log_\pi = \text{Log}$ on $\mathbb{C}_\pi$ ⟨p. 83 ∈ [1]⟩
- For $G \underset{area}{\subset} \mathbb{C}\backslash\{0\}$:
  If there *exists* a *continuous argument function* α for G, then
  $l(z) = \log|z| + i\alpha(z), z \in G$ is a *holomorphic branch* of the *logarithm*.
  $l(z)$ is *stamfunktion* for $\frac{1}{z}$, i.e.: $l'(z) = \frac{1}{z}$ for $z \in G$ ⟨thm. 5.14 p. 83 ∈ [1]⟩
- *Any* $G \underset{enkeltsammenhngende\ area}{\subset} \mathbb{C}\backslash\{0\}$ has a *continuous argument function* ⟨thm. 5.14 p. 83 ∈ [1]⟩
- For $G \underset{area}{\subset} \mathbb{C}\backslash\{0\}$:
  If $l(z)$ is a *holomorphic branch* of the *logarithm* on G, then for $p \in \mathbb{Z}$:
  $l(z) + 2\pi ip$ is *also* a *holomorphic branch* of the *logarithm* on G
  and *there are no others*. ⟨rem. 5.15 p. 84 ∈ [1]⟩
- For $|z| < 1$ we have the *power series development*:
  $\text{Log }(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots$ in $K(0, 1)$. ⟨thm. 5.16 p. 84 ∈ [1]⟩

9 Power

- For $\alpha, z \in \mathbb{C}$: $z^\alpha \overset{def}{=} \exp(\alpha \log z)$, and z^α is an *infinite set*.
  E.g.: $i^i = \{e^{-\frac{\pi}{2}-2\pi p} \mid p \in \mathbb{Z}\}$ ⟨p. 84 ∈ [1]⟩

- If l is a *holomorphic branch* of the *logarithm* for an *area* $G \underset{area}{\subset} \mathbb{C}\setminus\{0\}$ then:
  $\exp(\alpha l(z))$ is a *holomorphic branch* of z^α and
  $\frac{d}{dz} z^\alpha = \alpha l'(z) \exp(\alpha l(z)) = \frac{\alpha}{z} \exp(\alpha l(z)) = \alpha \exp((\alpha - 1) l(z))$ ⟨p. 84-85 ∈ [1]⟩
  Practical (but incorrect) notation:
  $\frac{d}{dz} z^\alpha = \alpha z^{\alpha - 1}$ (where z^α is considered *some holomorphic branch*)

- $z \mapsto \exp(\alpha \log(1+z))$ is a *holomorphic branch* of
  $(1+z)^\alpha$ in $\mathbb{C}\setminus\{z \in \mathbb{R} \mid z \leq 1\}$ ⟨p. 85 ∈ [1]⟩

- The *binomial series*: Let $\alpha \in \mathbb{C}$. For $|z| < 1$ we have:
  $(1+z)^\alpha = \exp(\alpha \log(1+z)) = \sum_{n=0}^{\infty} z^n$ for $z \in K(0,1)$ where
  $\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = 1$ and $\begin{pmatrix} \alpha \\ n \end{pmatrix} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ for $n \geq 1$ ⟨thm. 5.17 p. 85 ∈ [1]⟩

- For $G \underset{area}{\subset} \mathbb{C}$, $f \in \mathcal{H}(G), \forall z \in G : f(z) \neq 0$, and $f(G)$ is *contained* in an *enkeltsammenhængende area* $\Omega \subset \mathbb{C}\setminus\{0\}$ then:
  we can use $l \circ f$ and $\exp(\alpha l \circ f)$ as *holomorphic branches* of $\log f$ and f^α, for $\alpha \in \mathbb{C}$, where l is a *holomorphic logarithm* for Ω.
  Sometimes we can even find *holomorphic branches without* the condition on $f(G)$ ⟨p. 85 ∈ [1]⟩

- Riemann's mapping theorem: For $G \underset{enkeltsammenhngende\ area}{\subset} \mathbb{C}, G \neq \mathbb{C}$:
  there *exists* a *bijective holomorphic function* $\phi : G \to K(0,1)$.
  ϕ is also *biholomorphic*. ⟨thm. 5.19 p. 86 ∈ [1]⟩

- For $G \underset{area}{\subset} \mathbb{C}$ the *following are equivalent*:
  - G is *enkeltsammenhængende* ⟨thm. 5.18 p. 86 ∈ [1]⟩ ⟨Cauchy's Integral Thm. ∈ [1]⟩
  - For *any* $f \in \mathcal{H}(G)$ and *any closed contour* γ in G: $\int_\gamma f(z)dz = 0$
    ⟨thm. 5.18 p. 86, thm. 2.13 p. 38, Cauchy's Integral Thm. ∈ [1]⟩
  - Any $f \in \mathcal{H}(G)$ has a *stamfunktion* ⟨thm. 5.18 p. 86, thm. 2.13 p. 38 ∈ [1]⟩
  - For *any* $f \in \mathcal{H}(G)$ where $\forall z \in G : f(z) \neq 0$:
    f has a *holomorphic logarithm*, i.e. an $l \in \mathcal{H}(G)$ such that $\exp\ l = f$
    ⟨thm. 5.18 p. 86 ∈ [1]⟩
  - For *any* $f \in \mathcal{H}(G)$ where $\forall z \in G : f(z) \neq 0$:
    f has a *holomorphic square root*, i.e. an $g \in \mathcal{H}(G)$ such that $g^2 = f$
    ⟨thm. 5.18 p. 86 ∈ [1]⟩

10 A Holomorphic Branch of Arcsin

Premise: $G = \mathbb{C} \backslash \{x \in \mathbb{R} \mid |x| \geq 1\}$

- Arcsin $x = \int_{[0,x]} \frac{1}{\sqrt{1-u^2}} du$, for $x \in G$ ⟨ex. 5.21 p. 86-87 ∈ [1]⟩
- Arcsin$|_{]-1,1[}$ is the *inverse* if $\sin :]-\frac{\pi}{2}, \frac{\pi}{2}[\rightarrow]-1,1[$ ⟨ex. 5.21 p. 87 ∈ [1]⟩
- $\frac{d}{dx}$Arcsin $x = \frac{1}{\sqrt{1-x^2}}$ ⟨ex. 5.21 p. 87 ∈ [1]⟩
- $\frac{1}{\sqrt{1-x^2}}$ and Arcsin x are *holomorphic* in G ⟨ex. 5.21 p. 87 ∈ [1]⟩

11 Elliptic Functions

- For *polynomium* p with *degree* ≥ 1 and *roots* $r_1, r_2, \ldots, r_n$:
  $\frac{1}{p(z)}$ is *holomorphic* in $G = \mathbb{C} \backslash \{r_1, \ldots, r_n\}$, which is *not enkeltsammenhængende.*
  To find a *holomorphic branch* of $\frac{1}{p(z)}$ we have to look at an *enkeltsammenhængende subarea* of G by e.g. *removing some halflines* ⟨ex. 5.22 p. 87-88 ∈ [1]⟩
- Stamfunktioner: $F(z) = \int_{[z_0,z]} \frac{1}{\sqrt{p(u)}} du$
  (*integrating* some *contour* from z_0 to z) ⟨ex. 5.22 p. 87-88 ∈ [1]⟩
- For *polynomials* of *degree* 1 and 2, the *stamfunktioner* $F(z)$ can be expressed by the *logarithm function* and the *inverse trigonometric functions* ⟨ex. 5.22 p. 88 ∈ [1]⟩
- For *polynomials* of *degree* 3 and 4, the *stamfunktioner* $F(z)$ are called *elliptic integrals.*
  Their *inverse functions* are called the *elliptic functions* ⟨ex. 5.22 p. 88 ∈ [1]⟩
- Elliptic functions have 2 *linearly independent periods*,
  typically a *real period* and a *purely imaginary period* ⟨ex. 5.22 p. 88 ∈ [1]⟩

12 Zero Points and Isolated Singularities

- Algebra: $\mathcal{M}(G)$ is *brøklegemet* for *integritetsområdet* of $\mathcal{H}(G)$ ⟨p. 95 ∈ [1]⟩
- A *real (infinitely often) differentiable function* can be 0 in an *interval* without being the 0-function.
  This is *not possible* for *holomorphic functions*! ⟨p. 95 ∈ [1]⟩
- If a *polynomial* of *degree* ≥ 1 has a *zero point* (i.e. a *root*) a, i.e. $p(a) = 0$, then p can be *factored* into:
  $p(z) = (z - a)^m q(z)$ where q is a *polynomial* with $q(a) \neq 0$ and
  $m \geq 0$ is the *multiplicity / order* of the *root* ⟨p. 95 ∈ [1]⟩
- For $G \underset{area}{\subset} \mathbb{C}$, $f : G \to \mathbb{C}$ *holomorphic* and $f \neq 0$-function.
  If $a \in G$ is a *zero point* for f (i.e. $f(a) = 0$), then:
  there *exists* a *unique* $n \in \mathbb{N}$ and a *unique holomorphic function* $g \in \mathcal{H}(G)$ with $g(a) \neq 0$ such that:
  $f(z) = (z - a)^n g(z)$ for $z \in G$.
  n is the *multiplicity / order* of the *root* a: $\mathbf{ord}(f, a)$ ⟨thm. 6.1 p. 95 ∈ [1]⟩
- We say that for $f(a) \neq 0$, a has $\text{ord}(f, a) = 0$ ⟨p. 96 ∈ [1]⟩
- **A *simple zero point / simple root* a** $\overset{def}{=} \text{ord}(f, a) = 1$ ⟨p. 96 ∈ [1]⟩
- For $G \underset{area}{\subset} \mathbb{C}$, $f : G \to \mathbb{C}$ *holomorphic* and $f \neq 0$-function:
  For a *zero point* $a \in G$, $\text{ord}(f, a)$ is *characterized by*:
  $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$, $f^{(n)}(a) \neq 0$ ⟨cor. 6.2 p. 97 ∈ [1]⟩
- For $G \underset{area}{\subset} \mathbb{C}$, $f : G \to \mathbb{C}$ *holomorphic* and $f \neq 0$-function:
  Any *zero point* is *isolated* - i.e. $\exists r > 0 : \forall z \in K'(a, r) : f(z) \neq 0$.
  The *set* of *zero points* for f, $Z(f)$, is *discrete* in G,
  so in particular $Z(f)$ is *countable* ⟨thm. 6.3 p. 97 ∈ [1]⟩
- For $G \underset{area}{\subset} \mathbb{C}$, and *arbitrary discrete* set $P \subset G$,
  there *exists* $f \in \mathcal{H}(G)$ with $Z(f) = P$.
  It is even possible to *choose* f with *desired multiplicities* ⟨p. 97 ∈ [1]⟩
- For $G \underset{area}{\subset} \mathbb{C}$, $f : G \to \mathbb{C}$ *holomorphic*: ⟨p. 97 ∈ [1]⟩
  for *any* $\lambda \in \mathbb{C}$, the set $\{z \in G \mid f(z) = \lambda\}$ is *either* $\emptyset$, G or *countable*
- The *Identity* Theorem for *holomorphic functions*:
  For $G \underset{area}{\subset} \mathbb{C}$, $f, g : G \to \mathbb{C}$ *holomorphic*:
  if $f(z) = g(z)$ for *all* z in a *subset* $A \subset G$, which has a *condensation point* in G, then
  $\forall z \in G : f(z) = g(z)$ ⟨thm. 6.5 p. 97 ∈ [1]⟩
- For $G \underset{area}{\subset} \mathbb{C}$, $f, g \in \mathcal{H}(G)$, $f(a) = g(a) = 0$ for $a \in G$ and $f, g \neq 0$-function in a *neighborhood* of a: L'Hospital's rule: ⟨thm. 6.7 p. 98 ∈ [1]⟩
  The *limit* $\lim_{z \to a} \frac{f(z)}{g(z)}$ *exists* $\Leftrightarrow \text{ord}(f, a) \geq \text{ord}(g, a)$, and then
  $\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f^{(q)}(a)}{g^{(q)}(a)}$, where $q = \text{ord}(g, a)$, and hence $g^{(q)}(a) \neq 0$

13 Isolated Singularities

- For $G \underset{area}{\subset} \mathbb{C}$, $a \in G$:
  If $f \in \mathcal{H}(G \backslash \{a\})$, then a is an *isolated singularity*.
  If f can be given a *complex value* in a such that f becomes *holomorphic* in G, the *singularity* is called *cancellable*.
  If f has a *cancellable singularity* in $z = a$, the *value* of $f(a)$ must be $\lim_{z \to a} f(z)$ ⟨def. 6.8 p. 99 ∈ [1]⟩

- For $G \underset{area}{\subset} \mathbb{C}$, $f \in \mathcal{H}(G)$ and $f \neq 0$-function:
  $\frac{1}{f}$ is *holomorphic* in $G \backslash Z(f)$,
  and *all points* $a \in Z(f)$ are *isolated singularities* ⟨p. 99 ∈ [1]⟩

- For $G \underset{area}{\subset} \mathbb{C}$, $f \in \mathcal{H}(G \backslash \{a\})$:
  If f *limited* in $K'(a, r)$ for some $r > 0$, then:
  f has a *cancellable singularity* at a ⟨thm. 6.9 p. 99 ∈ [1]⟩

- For $G \underset{area}{\subset} \mathbb{C}$, $f \in \mathcal{H}(G \backslash \{a\})$:
  If f has a *non-cancellable singularity* at a, then:
  f *cannot have a limit* for $z \to a$ ⟨p. 99 ∈ [1]⟩

13.1 Isolated Singularities: Poles and Essential Singularities

Premises: $G \underset{area}{\subset} \mathbb{C}$, $f \in \mathcal{H}(\mathbb{C}\backslash\{a\})$.

- Let a be an *isolated singularity.*
  a is a *pole* of *order* $m \in \mathbb{N}$ for f $\overset{def}{=}$
  $(z-a)^m f(z)$ has a *limit value* for $z \to a$ and $\lim_{z\to a}(z-a)^m f(z) \neq 0$.
  ⟨def. 6.10 p. 100 ∈ [1]⟩

- The *order* of a *pole* is *uniquely determined.* ⟨p. 100 ∈ [1]⟩

- $\lim_{z\to a}|f(z)| = \infty$ for a *pole* a at f. ⟨p. 100 ∈ [1]⟩

- Let a be a *pole* of *order* m at a:
  $g(z) = \begin{cases} (z-a)^m f(z) & ,z \in G\backslash\{a\} \\ \lim_{z\to a}(z-a)^m f(z) & ,z = a \end{cases}$
  is *holomorphic* in G and has *power series*:
  $g(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ in the *greatest open disc* $K(a,\rho) \subset G$.
  I.e.: $f(z) = \frac{a_0}{(z-a)^m} + \frac{a_1}{(z-a)^{m-1}} + \cdots + \frac{a_{m-1}}{(z-a)} + \sum_{k=0}^{\infty} a_{m+k}(z-a)^k$.
  ⟨p. 100 ∈ [1]⟩

- Let a be a *pole* of *order* m at a:
  The *principal part* of f at a $\overset{def}{=}$
  $p(\frac{1}{z-a}) = \sum_{k=1}^{m} \frac{a_{m-k}}{(z-a)^k}$, where $p(z) = \sum_{k=1}^{m} a_{m-k} z^k$. ⟨p. 100 ∈ [1]⟩
  When *subtracting* the *principal part* of f from f,
  the *result* has a *cancellable singularity* at a.

- **An *essential* ⟨DK: *væsentlig*⟩ *singularity* a of f** $\overset{def}{=}$
  a is an *isolated singularity* which is *neither cancellable* nor a *pole.*
  ⟨p. 100 ∈ [1]⟩

- *Picard's big Theorem*: Let a be an *essential singularity.*
  For *any* $r > 0$ such that $K(a,r) \subset G$, the set $f(K'(a,r))$ is *either all* of $\mathbb{C}$ or $\mathbb{C}$ with *one point excluded.* ⟨p. 100 ∈ [1]⟩

- *Casiorati-Weierstrass's Theorem*: If f has an *essential singularity* at a, then $f(K'(a,r))$ is *everywhere dense* in $\mathbb{C}$ for *any* $r > 0$ where $K(a, r \subset G)$.
  ⟨thm. 6.11 p. 101 ∈ [1]⟩

14 Rational Functions

- **A *rational function* of a *complex variable* z $\overset{def}{=}$**
an *expression* of the form $\frac{p(z)}{q(z)}$, where $p, q \in \mathbb{C}[z]$, $q \neq 0$-function.
  - Notice: $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are considered the *same function* if $p_1 q_2 = p_2 q_1$.
  - Notice: If p and q have *common zero points* $z = a$, then a *suitable power* of $(z - a)$ can be *divided away*, so we can always *assume* that p and q does *not* have common zero points.

⟨p. 101 ∈ [1]⟩

- For $f(z) = \frac{p(z)}{q(z)}$ *rational function*:
f is *holomorphic* in $\mathbb{C} \backslash Z(q)$ and $Z(f) = Z(p)$. ⟨p. 101 ∈ [1]⟩

- For $f(z) = \frac{p(z)}{q(z)}$ *rational function*:
Let *q's zero points* be $a_1, \ldots, a_k$ with *multiplicities* $m_1, \ldots, m_k$, then:
$a(z) = c(z-a_1)^{m_1} \cdots (z-a_k)^{m_k}$ and $a_1, \ldots, a_k$ are *poles* of *order* $m_1, \ldots, m_k$.
⟨p. 101-102 ∈ [1]⟩

- Algebra: **Integritets området** $\overset{def}{=}$
a *commutative ring* where $ab = 0 \Rightarrow a = 0 \vee b = 0$. ⟨p. 102 ∈ [1]⟩

- Algebra: The *set* of *rational functions* is called $\mathbb{C}(z)$.
$\mathbb{C}(z)$ is a *commutative legeme*, *brøklegement* for *integritets området* $\mathbb{C}[z]$, since *any* element $p/q \in \mathbb{C}(z) \backslash \{0\}$ has a *reciprocal element* q/p. ⟨p. 102 ∈ [1]⟩

- For $f(z) = \frac{p(z)}{q(z)}$ *rational function* and $degree(p) \geq degree(q)$ we can *divide* and get $f(z) = q_1(z) + \frac{r(z)}{q(z)}$, for $p = q_1 q + r$ with $degree(r) < degree(q)$.
r and q will *not* have *common zero points*. ⟨p. 102 ∈ [1]⟩

- Decomposition: Let $r, q \in \mathbb{C}[z]$ be *without common zero points*,
$0 \leq degree(r) < degree(q)$ and let $a_1, \ldots, a_k$ be the *zero points* for q with *multiplicities* $m_1, \ldots, m_k$.
Then there *exists unique constants* $c_{j,l} \in \mathbb{C}$ such that:
$\frac{r(z)}{q(z)} = \sum_{j=1}^{k} \sum_{l=1}^{m_j} \frac{c_{j,l}}{(z-a_j)^l}$.
I.e.: *Rational functions* is a *sum* of it's *principal parts*. ⟨thm. 6.12 p. 102 ∈ [1]⟩
Methods to find $c_{j,l}$:
Proof p. 102-103 and bottom p. 103 or example 6.13 p. 104.

- $\mathbb{C}(z)$ is a *vector space* over $\mathbb{C}$ with *basis* consisting of:
The *monomials* $1, z, z^2, \ldots$ *and* the *functions* $\frac{1}{(z-a)^k}$, $a \in \mathbb{C}$, $k \in \{1, 2, \ldots\}$.
⟨rem. 6.14 p. 104 ∈ [1]⟩

15 Meromorphic Functions

- For $G \underset{area}{\subset} \mathbb{C}$: $h : G \to \mathbb{C} \cup \{\infty\}$ **is *meromorphic*** $\overset{def}{=}$
  1) $P = \{z \in G \mid h(z) = \infty\}$ is *discrete* in G
  2) The *restriction* $f = h|_{G\setminus P}$ is *holomorphic* in the *open set* $G\setminus P$
  3) *Any point* $a \in P$ is a *pole* for f

  I.e.: *Meromorphic* is *holomorphic* with *only cancellable singularities* and *poles* ⟨def. 6.15 p. 104 ∈ [1]⟩

- The *set* of *meromorphic functions* in $G \underset{area}{\subset} \mathbb{C}$ is: $\mathcal{M}(G)$ ⟨p. 104 ∈ [1]⟩

- A *holomorphic function* is *meromorphic* with $P = \emptyset$ ⟨p. 105 ∈ [1]⟩

- If $h(z) = \frac{f(z)}{g(z)}$ is a *quotient* of 2 *holomorphic functions* f, g, where $g \neq 0$-function, then: h is *meromorphic*.
  The *poles* of h are the *roots* of g whose *order* as *denominator points* are *higher* than the *order* as *noninator points* ⟨p. 105 ∈ [1]⟩

- A *rational function* is *meromorphic* in $\mathbb{C}$ with *finitely many poles* ⟨p. 105 ∈ [1]⟩

- Algebra: $\mathcal{M}(G)$ is a *commutative legeme.*
  It is *isomorphic* to *brøklegemet* for *integritetsområdet* $\mathcal{H}(G)$ ⟨p. 105 ∈ [1]⟩

16 Laurent Series

- **A *Laurent series*** $\stackrel{def}{=}$
  a *double infinite series* of the form: $\sum_{n=-\infty}^{\infty} c_n z^n$, where
  $(c_n)_{n\in\mathbb{Z}}$ are *given complex numbers* and $z \in \mathbb{C}\backslash\{0\}$ ⟨def. 6.16 p. 106 ∈ [1]⟩

- Convergence of *Laurent series*: $\sum_{n=-\infty}^{\infty} c_n z^n$ is considered the *sum* of 2 *usual series* $\sum_{n=1}^{\infty} c_{-n} z^{-n} + \sum_{n=0}^{\infty} c_n z^n$, *both* of which *must converge.*
  They are *power series* in $\frac{1}{z}$ and z respectively.
  Radius of convergence ρ_1, ρ_2,
  the *first absolute convergent* for $|\frac{1}{z}| < \rho_1$, the *second* for $|z| < \rho_2$.
  Only $\frac{1}{\rho_1} < \rho_2$ is *interesting*, since it gives that:
  $\sum_{n=-\infty}^{\infty} c_n z^n$ is *convergent* in the *ring area* $G = \{z \in \mathbb{C} \mid \frac{1}{\rho_1} < |z| < \rho_2\}$
  between two *concentric circles* with *center* 0 and *radii* $\frac{1}{\rho_1}$ and ρ_2.
  For $\rho_1 = \infty$ and $\rho_2 < \infty$: $G = K'(0, \rho_2)$
  For $\rho_1 = \infty$ and $\rho_2 = \infty$: $G = \mathbb{C}\backslash\{0\}$
  $\sum_{n=-\infty}^{\infty} c_n z^n$ is *holomorphic* in G.

- ***Laurent series* with *development point*** $a \in \mathbb{C}$ $\stackrel{def}{=}$ ⟨p. 167 ∈ [1]⟩
  $\sum_{n=-\infty}^{\infty} c_n (z-a)^n$.
  Holomorphic in a *ring area* with *center* a: $\{z \in \mathbb{C} \mid R_1 < |z-a| < R_2\}$

- For *holomorphic function* in the *ring area* $G = \{z \in \mathbb{C} \mid R_1 < |z-a| < R_2\}$ with $0 \le R_1 < R_2 \le \infty$:
  f is represented in G as the *sum* of a *unique Laurent series*:
  $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ for $z \in G$ and the *coefficients* are given by:
  $c_n = \frac{1}{2\pi i} \int_{\partial K(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz$, $n \in \mathbb{Z}$ where $r \in]R_1, R_2[$ is *arbitrary*
  ⟨thm. 6.17 p. 107 ∈ [1]⟩

- For *holomorphic function* f in a *ring area* $G = \{z \in \mathbb{C} \mid R_1 < |z-a| < R_2\}$
  we have 2 *holomorphic functions* given by *power series*:
  $f_i(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ in $K(a, R_2)$ (interior)
  $f_e(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}$ in $\{z \in \mathbb{C} \mid |z-a| > R_1\}$ (exterior)
  where $\forall z \in G : f(z) = f_i(z) + f_e(z)$ ⟨rem. 6.18 p. 110 ∈ [1]⟩

- For *holomorphic* f in the *ring area* $R_1 < |z-a| < R_2$:
  for $r \in]R_1, R_2[$ we consider the function:
  $g_r(\theta) = f(a + re^{i\theta})$, $\theta \in \mathbb{R}$, which is 2π-periodic. We have:
  $g_r(\theta) = \sum_{n=-\infty}^{\infty} c_n r^n e^{in\theta}$
  which is the *Fourier series* for g_r which *converges uniformly* for $\theta \in \mathbb{R}$.
  $c_n r^n = \frac{1}{2\pi} \int_0^{2\pi} g_r(\theta) e^{-in\theta} d\theta$, the *usual formula* for the *Fourier coefficients.*
  ⟨rem. 6.19 p. 110-111 ∈ [1]⟩

- For $G \underset{open}{\subset} \mathbb{C}$, $a \in G$, $f \in \mathcal{H}(G\backslash\{a\})$:
  f has an *isolated singularity* at a and a *Laurent series* in $K'(a, \rho)$,
  the *greatest open disc* in G. So $f(z) = f_i(z) + f_e(z)$ where
  $f_i(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ in $K(a, \rho)$
  $f_e(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}$ in $\mathbb{C}\backslash\{a\}$
  The *principal part* of f $\stackrel{def}{=}$
  $f_e(z)$ given by the *sum* for the *negative powers* ⟨def. 6.20 p. 111 ∈ [1]⟩

$f(z) - f_e(z)$ is *holomorphic* in $G\backslash\{a\}$ with a *cancellable singularity* at a, and $f_i(z)$ determines the *holomorphic extension* to a. ⟨p. 111 ∈ [1]⟩

- The *isolated singularity* a for $f \in \mathcal{H}(G\backslash\{a\})$ with *Laurent series* $\sum_{n=-\infty}^{\infty} c_n(z-a)^n$ is:
  1) *Cancellable* $\Leftrightarrow$ $c_n = 0$ for $n < 0$
  2) A *pole* $\Leftrightarrow$ $c_n = 0$ for all $n < 0$ *except finitely many*
  3) *Essential* $\Leftrightarrow$ $c_n \neq 0$ for *infinitely many* $n < 0$

  ⟨thm. 6.21 p. 112 ∈ [1]⟩

- For $P \underset{discrete}{\subset} \mathbb{C}$, $f \in \mathcal{H}(\mathbb{C}\backslash P)$ where *none of f's singularities* $a \in P$ *are cancellable*:
  $\rho = \inf\{|z_0 - a| \mid a \in P\}$ is *radius of convergence* for the *series*:
  $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ ⟨p. 114 ∈ [1]⟩

17 Residuals And Their Use

- For $G \underset{area}{\subset} \mathbb{C}$, $f \in \mathcal{H}(G \backslash \{a\})$, f has an *isolated singularity* for $z = a \in G$, $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$:
  The *residual* of f in a, $Res(f, a) \overset{def}{=} c_{-1}$. And we have:
  $Res(f, a) = c_{-1} = \frac{1}{2\pi i} \int_{\partial K(a,r)} f(z) dz$, for $0 < r < \rho$, where $K(a, \rho)$ is the *greatest circle* in G with *center* a.

- Cauchy's residual theorem: For $G \underset{enkeltsammenhngende\ area}{\subset} \mathbb{C}$, $P = \{a_1, \ldots, a_n\} \subset G$, γ a *simple closed contour* in G which *surrounds* $a_1, \ldots, a_n$ and is *traversed once counter clockwise (i.e. positive orientation)*, $f \in \mathcal{H}(G \backslash P)$:
  $\int_\gamma f(z) dz = 2\pi i \sum_{j=1}^{n} Res(f, a_j)$ ⟨thm. 7.1 p. 121 ∈ [1]⟩

- Some *special cases* of *residuals* for a *meromorphic function* $h(z)$:
  - If h has a *simple pole* in a, then:
    $Res(h, a) = \lim_{z \to a} (z-a) h(z)$ ⟨p. 122 ∈ [1]⟩
  - If $h = \frac{f}{g}$ has a *simple pole* in a and $f(a) \neq 0$, $g(a) = 0$, $g'(a) \neq 0$, then:
    $Res(h, a) = \frac{f(a)}{g'(a)}$ ⟨p. 123 ∈ [1]⟩
  - If h has a *pole* of *order* $m \geq 1$ in a, then if we *define* $\phi(z) = (z-a)^m h(z)$ (i.e. ϕ has *cancellable singularity* in a) we have:
    $Res(h, a) = \frac{\phi^{(m-1)}(a)}{(m-1)!}$ ⟨p. 123 ∈ [1]⟩

17.1 Concrete Examples of Resuduals

- The *residual* of the *rational* function $z \mapsto \frac{1}{(1+z^2)}$ in the points $z = \pm i$ is: $\mp \frac{i}{2}$. ⟨ex. 7.2 (i) p. 123 ∈ [1]⟩

- The *meromorphic* function $z \mapsto \frac{1}{\sin z}$ has *simple poles* for $z = n\pi$ for $n \in \mathbb{Z}$. The *residual* in $z = n\pi$ is $\frac{1}{\cos(n\pi)} = (-1)^n$. ⟨ex. 7.2 (ii) p. 123 ∈ [1]⟩

- The function $h(z) = \frac{z \sin z}{1 - \cos z}$ is *meromorphic* in $\mathbb{C}$.
  See book p. 123 for further analysis. ⟨ex. 7.2 (iii) p. 123 ∈ [1]⟩

- $h(z) = z^2 (z^2 + 1)^{-2}$ is a *rational* function with *poles* of *order* 2 in $z = \pm i$.
  $Res(h, i) = -\frac{i}{4}$ and $Res(h, -i) = \frac{i}{4}$. ⟨ex. 7.2 (iv) p. 124 ∈ [1]⟩

18 The Argument Principle

- For $G \underset{enkeltsammenhngende\ area}{\subset} \mathbb{C}$, $h : G \to \mathbb{C} \cup \{\infty\}$ *meromorphic* function, γ *positively oriented simple closed contour* in G which does *not pass any* of $h's$ *zero points* or *poles*:
  $\frac{1}{2\pi i} \int_\gamma \frac{h'(z)}{h(z)} dz$ equals:
  The number N of *zero points minus* the number P of *poles* for h in the *subarea* that γ *sorrounds.*
  Each *pole* or *zero point* of *order m counts* as the number m. ⟨thm. 7.3 p. 124 ∈ [1]⟩

- *Argument principle*:
  For $G \underset{enkeltsammenhngende\ area}{\subset} \mathbb{C}$, $h : G \to \mathbb{C} \cup \{\infty\}$ *meromorphic* function, γ *positively oriented simple closed contour* in G which does *not pass any* of $h's$ *zero points* or *poles*, $\Gamma = h \circ \gamma$ the *composite closed contour*:
  Notice:
  Γ is *contained* in $\mathbb{C} \backslash \{0\}$, since h *does not* have any *zero points* along γ^*.
  If γ is defined on $[a, b]$ then:
  $\frac{1}{2\pi i} \int_a^b \frac{\Gamma'(z)}{\Gamma(z)} dz = \frac{1}{2\pi i} \int_a^b \frac{h'(\gamma(t))\gamma'(t)}{h(\gamma(t))} dt = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z} dz = \omega(\Gamma, 0)$, so:
  $\omega(\Gamma, 0) = N - P$ and:
  $argvar(\Gamma) = 2\pi(N - P)$. ⟨thm. 7.4 p. 125-126 ∈ [1]⟩
  Where N: The number of *zero points*, P: the number of *poles* for h, counted with multiplicities.

- *Rouché's Theorem*: For $G \underset{enkeltsammenhngende\ area}{\subset} \mathbb{C}$, $f, g \in \mathcal{H}(G)$, γ *simple closed contour* in G:
  If $\forall z \in \gamma^* : |f(z) - g(z)| < |f(z)|$ then:
  f and g have the *same number* of *zero points counted with multiplicity* in the *limited area* that γ *sorrounds.* ⟨thm. 7.5 p. 126 ∈ [1]⟩

18.1 Topology Related

- *The Open Mapping Theorem*: For $G \underset{area}{\subset} \mathbb{C}$, f *non-constant holomorphic*: $f(G)$ is an *area* and f is an *open map* (i.e. Ω *open* $\Rightarrow f(\Omega)$ *open*).
  ⟨thm. 7.6 p. 127 ∈ [1]⟩

19 Calculating Definite Integrals

- Method: $\int_{\int}^{\int} f(x)dx$: *Choose closed contour* γ in $\mathbb{C}$ *containing* $[-R, R]$ (e.g. a *half circle* or a *rectangle*) and consider a *meromorphic* function which *equals* f on the *real axis.* The *integral* from $-R$ to R *plus* the *integral* over the rest of the contour is:
  $2\pi i$ *times* "*sum* of *residuals* of *poles contained* in γ".
  Usually, for $R \to \infty$ the contribution over "the rest of the contour" goes towards 0.
  - E.g.: *Half circle*: $\int_{-R}^{R} f(x)dx + \int_0^{\pi} \frac{R^2 e^{2it}}{(1+R^2e^{2it})^2} Rie^{it}dt$ is the "*sum* of *residuals* of *poles* in γ".
    The integral $\int_0^{\pi} \frac{R^2 e^{2it}}{(1+R^2e^{2it})^2} Rie^{it}dt$ goes towards 0 for $R \to \infty$.

  Theorems below gives *conditions* for *half circle* etc. ⟨p. 128 ∈ [1]⟩

- For f *rational*: If $f(z) = \frac{p(z)}{q(z)} = \frac{a_0+a_1z+\cdots+a_mz^m}{b_0+b_1z+\cdots+b_nz^n}$, $a_m \neq 0$, $b_n \neq 0$ and *assuming* $n \geq m+2$ and that f *does not* have any *poles* on the *real axis*: $\int_{\infty}^{\infty} f(x)dx = 2\pi i \sum_{j=1}^{k} Res(f, z_j)$, where $z_1, \ldots, z_k$ are the *poles* on the *upper half plane.* ⟨thm. 7.8 p. 129 ∈ [1]⟩

- For f *rational*: If $f(z) = \frac{p(z)}{q(z)} = \frac{a_0+a_1z+\cdots+a_mz^m}{b_0+b_1z+\cdots+b_nz^n}$, $a_m \neq 0$, $b_n \neq 0$ and *assuming* $n \geq m+2$ and that f *does not* have any *poles* on the *real axis*: $\int_{\infty}^{\infty} f(x)dx = -2\pi i \sum_{j=1}^{l} Res(f, \omega_j)$, where $\omega_1, \ldots, \omega_l$ are the *poles* on the *lower half plane.* ⟨rem. 7.9 p. 130 ∈ [1]⟩

- ***Fourier integrals*** $\stackrel{def}{=}$ integrals of the *form*: $\int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx$. ⟨p. 130 ∈ [1]⟩

- For f *meromorphic* in $\mathbb{C}$ *without poles* on the *real axis* and *only with finitely many poles* $z_1, \ldots, z_k$ in the *upper half plane*:
  If $\max_{0 \leq t \leq \pi} |f(Re^{it})| \to 0$ for $R \to \infty$ then:
  $\lim_{R\to\infty} \int_{-R}^{R} f(x)e^{i\lambda x}dx = 2\pi i \sum_{j=1}^{k} Res(f(z)e^{i\lambda z}, z_j)$, for $\lambda > 0$.
  ⟨thm. 7.10 p. 130 ∈ [1]⟩

- For f *meromorphic* in $\mathbb{C}$ *without poles* on the *real axis* and *only with finitely many poles* $\omega_1, \ldots, \omega_k$ in the *lower half plane*:
  If $\max_{\pi \leq t \leq 2\pi} |f(Re^{it})| \to 0$ for $R \to \infty$ then:
  $\lim_{R\to\infty} \int_{-R}^{R} f(x)e^{i\lambda x}dx = -2\pi i \sum_{j=1}^{l} Res(f(z)e^{i\lambda z}, \omega_j)$, for $\lambda < 0$.
  ⟨rem. 7.11 p. 131 ∈ [1]⟩

- Method: *Integrals* of the *form* $\int_0^{2\pi} f(\cos t, \sin t)dt$ can be *rewritten* to become a *curve integral.* ⟨p. 134 ∈ [1]⟩ FIXME: Add the details here...

19.1 Concrete Examples of Definite Integrals

- ⟨ex. 7.12 p. 131 $\in$ [1]⟩
- $\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x} dx = \pi$. ⟨ex. 7.13 p. 132-133 $\in$ [1]⟩
  This is a large example involving integration along a rectangle.
- ⟨ex. 7.14 p. 134 $\in$ [1]⟩

20 Evaluation of Infinite Series

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21 The Maximum Principle

- Local Maximum Principle: ⟨thm. 8.1 p. 143 ∈ [1]⟩
  For $G \underset{area}{\subset} \mathbb{C}$ and $f : G \to \mathbb{C}$ a *non-constant holomorphic* function:
  Then $|f|$ does *not have a local maximum* in *any point* $a \in G$.

- Global Maximum Principle: ⟨thm. 8.2 p. 144 ∈ [1]⟩
  For $G \underset{limited\ area}{\subset} \mathbb{C}$ and $f : \overline{G} \to \mathbb{C}$ *continuous* and *holomorphic* in G:
  The *maximum value* $M = \sup\{|f(z)| \mid z \in \overline{G}\}$ is assumed at a point at the *edge* of G, but *not* in *any point* of G *unless is constant.*

- For *holomorphic function* $f : K(0, \rho) \to \mathbb{C}$, where $0 < \rho \leq \infty$:
  The *maximum modulo* $m_f : [0, \rho[\to [0, \infty[\overset{def}{=}$
  $m_f(r) = \max\{f(z) \mid |z| < r\}$, $0 \leq r < \rho$.
  m_f is *growing*, and even *strictly growing* if f is *not constant.* ⟨p. 144 ∈ [1]⟩

- Schwartz' lemma: For $f : K(0, 1) \to K(0, 1)$ *holomorphic* with $f(0) = 0$:

  - $|f(z)| \leq |z|$, for $|z| < 1$
  - $|f'(0)| \leq 1$

  If one of these two inequalities have *equality*, then
  there *exists* a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that f is given by $f(z) = \lambda z$
  I.e. f is a *rotation* with *angle* $\arg(\lambda)$ around 0. ⟨thm. 8.3 p. 144 ∈ [1]⟩

- For $z_0 \in K(0, 1)$:
  $f_{z_0}(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$ is a *bijective holomorphic* function of $K(0, 1)$ onto $K(0, 1)$.
  The *inverse* is f_{-z_0} ⟨thm. 8.4 p. 146 ∈ [1]⟩

- For $G \underset{area}{\subset} \mathbb{C}$:
  $\mathrm{Aut}(G)$ is the *set of bijective holomorphic functions* $f : G \to G$.
  It is a *subgroup* of the *group of bijective functions* of G onto G.
  $\mathrm{Aut}(G)$ is called the *automophy group* for G. ⟨p. 146 ∈ [1]⟩

- $\mathrm{Aut}(K(0, 1)) = \{\lambda f_{z_0} \mid |\lambda| = 1, |z| < 1\}$, where $f_{z_0}(z) = \frac{z - z_0}{1 - \bar{z}_0 z}$. ⟨p. 146 ∈ [1]⟩

22 Theorems from Metric Spaces and Topology

22.1 Misc

- Let $G \underset{open}{\subset} \mathbb{C}$ and $\overline{K(a,r)} \subset G$. Then:
  there *exists* $R > r$ with $K(a,R) \subset G$ ⟨lemma 3.6 p. 47 ∈ [1]⟩
- *Borel's Coverage Theorem*: $K \underset{closed,limited}{\subset} \mathbb{C}$ or $K \underset{closed,limited}{\subset} \mathbb{R}^k$.
  For *any family* $(G_i)_{i \in I}$ of *open sets* in $\mathbb{R}^k$ which *covers* K
  (i.e. $K \subset \cup_{i \in I} G_i$),
  there *exists finitely many indices* $i_1, \ldots, i_n \in I$ such that
  $K \subset G_{i_1} \cup \cdots \cup G_{i_n}$. ⟨thm. 4.13 p. 64 ∈ [1]⟩
- If $K, F \underset{closed}{\subset} \mathbb{C}$ (or $\mathbb{R}^k$), $F, K \neq \emptyset$ and *disjoint*, K *limited* then:
  $d(K,F) = \inf\{|x - y| \mid x \in K, y \in F\} > 0$,
  and there *exists points* $x' \in K$ and $y' \in F$ such that $|x' - y'| = d(K,F)$
  ⟨thm. A.1 p. 149 ∈ [1]⟩
- If $K, F \underset{closed}{\subset} \mathbb{C}$ (or $\mathbb{R}^k$), $F, K \neq \emptyset$ and *disjoint*, F, K both *unlimited* then:
  it *may* be that $d(K,F) = 0$ ⟨thm. A.2 p. 149 ∈ [1]⟩
- $A \underset{closed}{\subset} \mathbb{R}^k$, A *limited*, $f : A \to \mathbb{R}^l$ *continuous* $\Rightarrow$
  $f(A)$ *limited* and *closed* in $\mathbb{R}^l$ ⟨thm. A.3 p. 150 ∈ [1]⟩
- $A \subset \mathbb{C}$ (or $\mathbb{R}^k$). $a \in \mathbb{C}$ **(or $\mathbb{R}^k$) is a *condensation point* for** $A \overset{def}{=}$
  $\forall r > 0 : K'(a,r) \cap A \neq \emptyset$ ⟨def. 5.4 p. 75 ∈ [1]⟩
  $\equiv$ there *exists* a *sequence* $x_n \in A \setminus \{a\}$ such that $x_n \to a$ for $n \to \infty$
- For $A \subset \mathbb{C}$ (or $\mathbb{R}^k$). $a \in A$ **is *isolated* in** $A \overset{def}{=}$
  a is *not* a *condensation point*, i.e.:
  $\exists r > 0 : K(a,r) \cap A = \{a\}$ ⟨def. 5.4 p. 75 ∈ [1]⟩
- $G \underset{open}{\subset} \mathbb{C}$. $A \subset G$ **is *discrete*** $\overset{def}{=}$
  A does *not* have *any condensation points* in G ⟨def. 5.5. p. 75 ∈ [1]⟩
- $G \underset{open}{\subset} \mathbb{C}$. $A \subset G$ is *discrete* $\Rightarrow$
  A *countable* and G A is *open* (i.e. A *closed* w.r.t. G) ⟨thm. 5.6 p. 75 ∈ [1]⟩
- $A \subset \mathbb{C}$ *curve connected*, $f : A \to \mathbb{C}$ *continuous*. $f(A)$ *discrete* in $\mathbb{C} \Rightarrow$
  f *constant* ⟨thm. 5.7 p. 76 ∈ [1]⟩
- **The *distance* from a *point* z to a *closed, non-empty set* F** $\overset{def}{=}$
  $d_F(z) = \inf\{|z - a| \mid a \in F\}$. ⟨p. 150 ∈ [1]⟩
  $d_F : \mathbb{C} \to [0, \infty[$ is *continuous* and:
  $\forall z \in \mathbb{C} : d_F(z) = 0 \Leftrightarrow z \in F$. ⟨thm. A.4 p. 150 ∈ [1]⟩
  Also holds in $\mathbb{R}^k$.
- For *any* $G \underset{open}{\subset} \mathbb{C}$ (or $\mathbb{R}^k$): There *exists* a *growing sequence* $K_1 \subset K_2 \subset \cdots$
  of *closed limited subsets* of G, where $\cup_{i \in \mathbb{N}} K_i = G$ ⟨thm. A.5 p. 151 ∈ [1]⟩
- For $A \subset \mathbb{C}$ *curve connected* and $f : A \to \mathbb{C}$ *continuous*:
  the *set* $f(A)$ is also *curve connected*. ⟨thm. A.10 p. 152 ∈ [1]⟩

22.2 Sequences and Convergence

- For *set* $M \subset \mathbb{C}$: **A *sequence of functions* $f_n : M \to \mathbb{C}$ *converges pointwise* towards** $f : M \to \mathbb{C} \overset{def}{=}$
  $\forall x \in M : \lim_{n\to\infty} f_n(x) = f(x) \equiv$ ⟨p. 55 ∈ [1]⟩
  $\forall x \in M : \forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \in \mathbb{N} : (n \geq N \Rightarrow |f(x) - f_n(x)| \leq \epsilon)$

- For *set* $M \subset \mathbb{C}$: **A *sequence of functions* $f_n : M \to \mathbb{C}$ *converges uniformly* towards** $f : M \to \mathbb{C} \overset{def}{=}$
  $\lim_{n\to\infty} \sup\{|f(x) - f_n(x)| \mid x \in M\} \to 0 \equiv$ ⟨def. 4.1 p. 56-57 ∈ [1]⟩
  $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \in \mathbb{N} : \sup\{|f(x) - f_n(x)| \mid x \in M\} \leq \epsilon \equiv$
  $\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \in \mathbb{N} : \forall x \in M : (n \geq N \Rightarrow |f(x) - f_n(x)| \leq \epsilon)$

- For $M \subset \mathbb{C}$ (or M *metric space*), and a *uniformly convergent sequence* $f_n : M \to \mathbb{C}$ *converging towards* $f : M \to \mathbb{C}$: ⟨thm. 4.2 p. 57 ∈ [1]⟩
  $\forall z_0 \in M : (\forall n \in \mathbb{N} : f_n$ *continuous* in $z_0) \Rightarrow f$ *continuous* in z_0

- The *sequence* $f_n : M \to \mathbb{C}$ *converges uniformly* towards $f : M \to \mathbb{C} \Rightarrow$
  $\forall A \subset M : f_n|_A$ *converges uniformly* towards $f|_A$ ⟨rem. 4.7 p. 59 ∈ [1]⟩

- If $f_n : M \to \mathbb{C}$ *converges pointwise* towards $f : M \to \mathbb{C}$, then:
  for *any* $A, B \subset \mathbb{C}$, where $f_n|_A$ *converges uniformly* towards $f|_A$ and
  $f_n|_B$ *converges uniformly* towards $f|_B$:
  $f_n|_{A\cup B}$ *converges uniformly* towards $f|_{A\cup B}$ ⟨rem. 4.7 p. 59 ∈ [1]⟩

22.3 Series and Convergence

- **An *infinite series* $\sum_{n=0}^{\infty} f_n(x)$ of *functions* $f_n : M \to \mathbb{C}$ is *uniformly convergent* with *stamfunktion*** $S : M \to \mathbb{C} \overset{def}{=}$
  The *prefix sequence* $S_n(x) = \sum_{k=0}^{n} f_k(x)$, for $x \in M$,
  converges uniformly towards S. ⟨def. 4.3 p. 58 ∈ [1]⟩

- *The Majorant Theorem of Weierstrass*: Let $\sum_{n=0}^{\infty} f_n(x)$ be an *infinite series* of *functions* $f_n : M \to \mathbb{C}$ and *assume* that there *exists* a *convergent majorant series*, i.e. a *convergent series* $\sum_{n=0}^{\infty} a_n$ where
  $\forall n \in \mathbb{N}_0 : a_n \geq 0$ and $\forall n \in \mathbb{N}_0 : \forall x \in M : |f(x)| \leq a_n$, then:
  $\sum_{n=0}^{\infty} f_n(x)$ *converges uniformly* on M. ⟨thm. 4.4 p. 58 ∈ [1]⟩

References

[1] Christian Berg. *Kompleks Funktionsteori*, Matematisk Institut Københavns Universitet 2002.